

UNIVERSAL POLYNOMIALS FOR TAUTOLOGICAL INTEGRALS ON HILBERT SCHEMES

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ABSTRACT. We show that tautological integrals on Hilbert schemes of points can be written in terms of universal polynomials in Chern numbers.

The results hold in all dimensions, though they strengthen known results even for surfaces by allowing integrals over arbitrary “geometric” subsets (and their Chern-Schwartz-MacPherson classes).

We apply this to enumerative questions, proving a generalised Göttsche Conjecture for all singularity types and in all dimensions. So if L is a sufficiently ample line bundle on a smooth variety X , in a general subsystem $\mathbb{P}^d \subset |L|$ of appropriate dimension the number of hypersurfaces with given singularity types is a polynomial in the Chern numbers of (X, L) .

When X is a surface, we get similar results for the locus of curves with fixed “BPS spectrum” in the sense of stable pairs theory.

1. INTRODUCTION

1.1. Main Results. Let X be a smooth, projective, connected variety of dimension d , and let E be an algebraic vector bundle on X . Denote by $X^{[n]}$ the Hilbert scheme of length n subschemes of X , and let $E^{[n]}$ be the tautological bundle on $X^{[n]}$ with fibre $H^0(Z, E|_Z)$ at $Z \in X^{[n]}$.

We study integrals of products of Chern classes of bundles $E^{[n]}$ over what we call *geometric subsets* of $X^{[n]}$. Geometric subsets form a natural class of subsets definable without reference to the global geometry of $X^{[n]}$. Specifically, the geometric subsets of $X^{[n]}$ are those generated by finite unions, intersections and complements from sets of the kind

$$\{Z \in X^{[n]} \mid Z = Z_1 \sqcup \dots \sqcup Z_k, \text{ } Z_i \text{ is of type } Q_i\}.$$

Here each Q_i is a constructible subset of the punctual Hilbert scheme $\text{Hilb}_0^{n_i}(\mathbb{C}^d) \subset \text{Hilb}^{n_i}(\mathbb{C}^d)$ of subschemes supported at $0 \in \mathbb{C}^d$, and we have $\sum n_i = n$. By “ Z_i is of type Q_i ” we mean that Z_i is isomorphic, as an abstract scheme, to an element of Q_i , so we require that the i -th connected component of Z is of isomorphism type in a specified family Q_i .¹

A k -variable Chern polynomial is a polynomial in the formal variables $c_i^{(j)}$, where $i \geq 1$ and $1 \leq j \leq k$. We treat such a Chern polynomial as a function from k -tuples of vector bundles to cohomology by the rule

$$c_i^{(j)}(E_1, \dots, E_k) = c_i(E_j),$$

extended linearly and multiplicatively to all Chern polynomials.

¹Clearly, a geometric subset $P \subseteq X^{[n]}$ is constructible and has the property that if $Z \in P$, $Z' \in X^{[n]}$ and $Z \cong Z'$, then $Z' \in P$. The converse is not quite true. For the basic subsets generating the algebra of geometric subsets we instead impose a similar condition on each connected component of Z , and this is a stronger requirement than the one above.

As an example of a subset with the above property which is not geometric by our definition, consider the following set. Let X be a surface, and let $P \subset X^{[n]}$ be the set containing all $Z = Z_1 \sqcup Z_2$ such that (1) each Z_i is defined by an ideal (\mathfrak{m}_i^5, f_i) where f_i is a product of 4 distinct linear factors, and (2) the cross ratio of the factors of f_1 equals that of the factors of f_2 .

A Chern monomial is a monomial in the variables $c_i^{(j)}$. The *weight* of a Chern monomial $c_{i_1}^{(j_1)} \cdots c_{i_k}^{(j_k)}$ is defined to be $\sum_{l=1}^k i_l$, so that treating a Chern monomial of weight l as a function, its image will be in $H^{2l}(X)$. Denote by $CM(k, l)$ the set of k -variable Chern monomials of weight l .

If Y is a proper scheme and $Z \subset Y$ a constructible subset, we denote by $c_{SM}(Z) \in H_*(Y)$ the Chern-Schwartz-MacPherson class of the characteristic function of Y . The construction and basic properties of this class are reviewed in Section 2.

The bulk of the paper is devoted to the proof of the following theorem.

Theorem 1.1. *Let X be a smooth, projective, connected variety of dimension d , let E_1, \dots, E_k be algebraic vector bundles on X , and let F be a k -variable Chern polynomial. Let either*

- (i) $N = \deg(F(E_1, \dots, E_k) \cap [P])$, for $P \subseteq X^{[n]}$ closed and geometric, or
- (ii) $N = \deg(F(E_1, \dots, E_k) \cap c_{SM}(P))$, for $P \subseteq X^{[n]}$ geometric.

Then, there exists a polynomial G in the variables $\{x_M\}_{M \in CM(k+1, d)}$, depending only on F , the ranks of the E_i , and the type of P , such that if we let x_M be the Chern number $\deg M(T_X, (E_i)) \cap [X]$, we have $G((x_M)) = N$.

If every point $Z \in P$ represents a subscheme with support in at most m points, the degree of G is at most m .

In case X is a surface or a curve, so that the Hilbert scheme is nonsingular, we can extend this as follows.

Theorem 1.2. *If $\dim X = 1$ or 2 , Theorem 1.1 holds with $F(E_1, \dots, E_k)$ replaced by $F(T_{X^{[n]}}, E_1, \dots, E_k)$.*

The strategy of the proof of the main theorem is motivated by J. Li's paper [Li], where he shows that the degree of the virtual fundamental class on the Hilbert scheme of points on a threefold is given by a universal polynomial. The outer structure of that proof, i.e. using the scheme $X^{[n]}$ and approximating by classes defined via $X^{[\alpha]}$, is well suited to our problem. Dealing with geometric subsets, the tautological bundles $E^{[n]}$ and the Chern-Schwartz-MacPherson class requires new ingredients.

An outline of the proof of the main theorem is given in Section 3, and the formal proof occupies Sections 4 and 5.

In Section 6, we show that a generating function for the Chern integrals of Theorems 1.1 and 1.2 (i) can be given a certain product form, a fact which is formally equivalent to the statement that part (i) of the theorems holds also for disconnected X .

A special case of Theorem 1.2 has been proved by Ellingsrud, Göttsche and Lehn using a completely different method, see [EGL]. In our terminology, they treat the case where X is a surface and the geometric subset P is the whole of $X^{[n]}$.

We note that the method of [EGL] yields a recursion which computes the integrals one are interested in. In contrast, our method is nonconstructive and relies at a crucial point on the fact that an element in the cohomology ring of a Grassmannian is a polynomial in the Chern classes of the universal bundle. Lacking a method of obtaining information about this polynomial, there is no apparent way of turning our proof into an algorithm.

1.2. Enumerative Applications. The main motivation for our result is to generalize the result known as the Göttsche Conjecture, which by now has several proofs, see [Ka], [KST], [Liu], and [Tz]. We recall the statement of the conjecture. Fix a surface S with a line bundle L which is "sufficiently ample", e.g. $L = M^{\otimes N}$, where M is a very ample line bundle and N is a sufficiently large integer. The precise definition of sufficiently ample uses the concept of N -very ampleness, see Section 7.

Let δ be a positive integer, and call a curve δ -nodal if it has δ nodes and no other singularities. If L is sufficiently ample, the locus of δ -nodal curves in $|L|$ has the expected codimension δ , so that in a general linear subsystem $\mathbb{P}^\delta \subset |L|$ there is a finite number of δ -nodal curves. The simplest form of the conjecture is then that there exists a degree δ polynomial N_δ in 4 variables, independent of S and L , such that the number of δ -nodal curves equals

$$N_\delta \left(c_1(L)^2, c_1(L)c_1(S), c_1(S)^2, c_2(S) \right).$$

Our main application is the generalization of this result to the case of curves with more general specified singularity types. Our approach follows the idea of Göttsche used in [Gö, Sec. 5] to reduce the problem of counting nodal curves to an integral on the Hilbert scheme. He defines a closed subset $W \subseteq S^{[3\delta]}$ and shows that the number of δ -nodal curves in the linear system \mathbb{P}^δ equals the degree of

$$c_{2\delta} \left(L^{[3\delta]} \right) \cap [W],$$

assuming L is $(5\delta - 1)$ -very ample. This idea was used by Tzeng in her proof of the Göttsche Conjecture in [Tz], which uses degenerations of S to show that the degree of the class above is a polynomial in the Chern numbers of (S, L) .

The set W appearing above is geometric, hence our theorem yields a different proof of Tzeng's result. Since our main theorem deals with more general loci in the Hilbert scheme of points, we may generalize the statement of Tzeng's theorem, replacing δ -nodal curves with curves having specified singularity types. Specifically, we have the following proposition.

Proposition (7.2). *Let S be a smooth, projective, connected surface, let L be a line bundle on S , and let T_1, \dots, T_k be analytic singularity types. Suppose L is N -very ample, where N is an integer depending on the types T_i , and let d be the sum of the codimensions of the T_i . There is a rational polynomial G of degree k in 4 variables, depending only on the T_i , such that in a general $\mathbb{P}^d \subseteq |L|$ the number of curves having precisely k singularities of types T_i is*

$$G(c_1^2(L), c_1(L)c_1(S), c_2(S), c_1^2(S)).$$

The same statement holds when the T_i are topological types.

Fixing types T_i , we may collect the universal polynomials for the number of curves having m_i singularities of type T_i in a generating function; this can then be written in the form $B_1^{c_1^2(L)} B_2^{c_1(L)c_1(S)} B_3^{c_1^2(S)} B_4^{c_2(S)}$ for rational power series B_i . See Corollary 7.3 for a precise statement.

Both of these results have recently been obtained independently by Li and Tzeng using a generalization of Tzeng's degeneration approach, see [LT].

By the same method we are able to count hypersurface singularities in arbitrary dimensions. Namely, we have the following.

Proposition (7.9). *Let X be a smooth, projective, connected variety, let L be a line bundle on X , and let T_1, \dots, T_k be analytic singularity types. Suppose L is N -very ample, where N is an integer depending on the types T_i , and let d be the sum of the codimensions of the T_i . There is a rational polynomial G in the Chern numbers of (X, L) , depending only on the T_i , such that in a general $\mathbb{P}^d \subseteq |L|$ the number of divisors having precisely k isolated singularities of types T_i is given by G .*

A different application of the main result concerns the locus of curves in a $\mathbb{P}^k \subset |L|$ having given "BPS spectrum". For a reduced, complete, locally planar curve C

with arithmetic genus $g(C)$ and geometric genus $\bar{g}(C)$, we consider the generating function

$$H_C(q) := \sum_{k=0}^{\infty} \chi(C^{[k]}) q^k$$

It is shown in [PT] that there are integers $n_{i,C}$ for $i = \bar{g}(C), \dots, g(C)$, such that

$$H_C(q) = \sum_{i=\bar{g}(C)}^{g(C)} n_{i,C} q^{g-i} (1-q)^{2i-2}.$$

If C is smooth, we have $H_C(q) = (1-q)^{2g-2}$, so this result can be interpreted as saying that in general $H_C(q)$ decomposes as a sum of $n_{i,C}$ copies of $q^{g-i} H_{C_i}(q)$ where C_i is smooth of genus i . We define $m_{i,C} = n_{g(C)-i,C}$, and it is then easy to check that the sequence of integers $(m_{i,C})_{i=0}^{\infty}$ depends only on the analytic type of the singularities of C . We refer to the sequence $(m_{i,C})$ as the *BPS spectrum* of C .

We note that the BPS spectrum is conjecturally determined by the Milnor numbers and links of the singularities of C , see [OS], and should therefore be a coarser invariant than topological singularity type. We show the following proposition.

Proposition (7.10). *Let S be a smooth, projective, connected surface, and let L be a line bundle on S . Let $m = (m_i)_{i=0}^{\infty}$ be a BPS spectrum, and denote by $|L|_m \subseteq |L|$ the locus of curves with BPS spectrum m . Let k be a nonnegative integer, let $\mathbb{P}^k \subseteq |L|$ be a general linear subsystem, and suppose L is N -very ample, where N is some integer depending on k and m . Then, there exists a rational polynomial G in 4 variables, depending only on k and m , such that*

$$\chi(\mathbb{P}^k \cap |L|_m) = G(c_1(L)^2, c_1(L)c_1(S), c_1(S)^2, c_2(S)).$$

This generalizes part of Kool, Shende and Thomas' proof of the Göttsche Conjecture in [KST], which involves the special case of the above proposition where m is the spectrum of a δ -nodal curve; that is $m = (m_i)$ with $m_i = \binom{\delta}{i}$.

1.3. Conventions. Homology and cohomology is singular with coefficients in \mathbb{Q} . By the degree of a class in $H_*(X)$ we mean its pushforward to $H_*(\text{pt}) \cong \mathbb{Q}$. In dealing with algebraic subsets of Hilbert schemes we always give these the reduced scheme structure, and few functors are represented. We often leave notation for pullback of vector bundles and cohomology classes implicit. We use the notation $\text{Gr}(r, N)$ for the Grassmannian parametrizing r -dimensional subspaces of \mathbb{C}^N .

If m is some number defined in terms of the data $X, (E_i), P, F$ of the theorem, we will use the shorthand “ m is universal” to mean that there exists a polynomial in the variables x_M computing m , depending only on F and the type of P as in the main theorem.

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2. PRELIMINARIES

Let X be a smooth, projective, connected variety of dimension d , and let E be an algebraic vector bundle on X . We give the definition of the tautological bundle $E^{[n]}$ and recall the construction of the Chern-Mather and Chern-Schwartz-MacPherson (abbreviated CSM) classes. In Section 2.3, we introduce the scheme $X^{[n]}$ and discuss the notion of geometric subsets of $X^{[n]}$ and $X^{[n]}$.

Denote by $\mathcal{Z} \subset X^{[n]} \times X$ the universal subscheme over $X^{[n]}$, and let $p : \mathcal{Z} \rightarrow X$ and $q : \mathcal{Z} \rightarrow X^{[n]}$ be the projections. The *tautological bundle* $E^{[n]}$ on $X^{[n]}$ is defined as

$$E^{[n]} = q_*(p^*(E)).$$

The flatness of q implies that $E^{[n]}$ is locally free, and we see that the fibre of $E^{[n]}$ at a point $Z \in X^{[n]}$ is the vector space $H^0(Z, E|_Z)$.

We next review the Chern-Mather and Chern-Schwartz-MacPherson classes. Both of these classes are generalizations to singular varieties of the Poincaré dual of $c_\bullet(T_X)$ for a smooth, proper X ; so for such X we have

$$c_{\text{SM}}(X) = c_M(X) = c_\bullet(T_X) \cap [X].$$

The Chern-Mather class is used in the definition of the CSM class and is essential in the proof of the main theorem.

2.1. Chern-Mather Class. Let Y be a reduced and irreducible projective scheme. We recall the definition of the Chern-Mather class $c_M(Y) \in H_*(Y)$.

The first step is to construct the Nash blow up $\tilde{Y} \rightarrow Y$. Suppose for a moment that Y is a reduced, irreducible and affine of dimension d . Let Y_{ns} be the nonsingular part of Y , and fix an embedding $f : Y \rightarrow \mathbb{A}^N$. The tangent map $T_{Y_{\text{ns}}} \rightarrow f^*(T_{\mathbb{A}^N})$ induces a morphism $g : Y_{\text{ns}} \rightarrow \text{Gr}(d, N)$, and we take \tilde{Y} to be the closure of the graph $\Gamma_g \subset Y \times \text{Gr}(d, N)$. The morphism $\tilde{Y} \rightarrow Y$ is defined by the projection $Y \times \text{Gr}(d, N) \rightarrow Y$, and we define the rank d vector bundle $T_{\tilde{Y}}$ on \tilde{Y} by restricting the universal bundle on $\text{Gr}(d, N)$.

It can be shown that this construction is independent of the choice of affine embedding and globalizes so that for any reduced, equidimensional scheme Y we get a well defined Y -scheme \tilde{Y} with a bundle $T_{\tilde{Y}}$. The morphism $\tilde{Y} \rightarrow Y$ is the *Nash blow up* of Y and the bundle $T_{\tilde{Y}}$ is the *Nash bundle*.

Definition 2.1. The Chern-Mather class $c_M(Y) \in H_*(Y)$ is the pushdown of $c_\bullet(T_{\tilde{Y}}) \cap [\tilde{Y}]$ along $\tilde{Y} \rightarrow Y$.

2.2. Chern-Schwartz-MacPherson Class. We recall the definition and basic properties of the Chern-Schwartz-MacPherson class. For details see [Fu, Ex. 19.1.7] and [Ma].

Let Y be a projective scheme, let $Z_*(Y)$ be the group of all cycles on Y , and let $F_*(Y)$ denote the group of constructible functions, where a function $f : Y \rightarrow \mathbb{Z}$ is called constructible if there exists a finite partition of Y into constructible sets such that f is constant on each stratum. Given any reduced scheme V , the local Euler obstruction $\nu_V : V \rightarrow \mathbb{Z}$ is a canonical constructible function determined at a point $x \in V$ by the analytic-local structure of V at x .²

There is a group homomorphism $\Omega : Z_*(Y) \rightarrow F_*(Y)$ defined on a primitive cycle V by

$$\Omega(V)(x) = \begin{cases} \nu_V(x) & \text{if } x \in V \\ 0 & \text{if } x \notin V \end{cases}$$

The map Ω is easily seen to be an isomorphism, using the fact that $\nu_V(x) = 1$ if $x \in V$ is a nonsingular point. The Chern-Mather class defines a homomorphism $c : Z_*(Y) \rightarrow H_*(Y)$ by letting

$$c(V) = i_*(c_M(V)),$$

where $i : V \rightarrow Y$ is the inclusion of a primitive cycle.

The Chern-Schwartz-MacPherson class $c_{\text{SM}}(f)$ is now defined for any constructible function f by

$$c_{\text{SM}}(f) = c(\Omega^{-1}(f)).$$

²If $p : \tilde{V} \rightarrow V$ is the Nash blow-up, we have $\nu_V(x) = \deg \left(c_\bullet \left(T_{\tilde{V}}|_{p^{-1}(x)} \right) \cap s(p^{-1}(x), \tilde{V}) \right)$, where $p^{-1}(x)$ is the scheme-theoretic inverse image and $s(-, -)$ denotes the Segre class.

It is clear that c_{SM} is a homomorphism. If $Z \subset Y$ is a constructible subset, we write $c_{\text{SM}}(Z) = c_{\text{SM}}(1_Z)$, where 1_Z is the characteristic function of Z . Additivity of c_{SM} then translates to

$$c_{\text{SM}}(Z_1 \cup Z_2) = c_{\text{SM}}(Z_1) + c_{\text{SM}}(Z_2) - c_{\text{SM}}(Z_1 \cap Z_2).$$

Given a morphism of proper schemes $g : Y_1 \rightarrow Y_2$, one can define a homomorphism $g_* : F(Y_1) \rightarrow F(Y_2)$ by letting

$$g_*(1_V)(x) = \chi(g^{-1}(x) \cap [V]) \quad x \in Y_2,$$

where $V \subset Y_1$ is a primitive cycle, and χ is the topological Euler characteristic. The main property of CSM classes, shown in [Ma], is that if g is proper, we have $g_*(c_{\text{SM}}(f)) = c_{\text{SM}}(g_*(f))$. As a corollary, letting Y be proper and g be the map to a point, we get that $\deg c_{\text{SM}}(Y) = \chi(Y)$.

2.3. Geometric Subsets. Following [Li], we introduce the scheme $X^{\llbracket n \rrbracket}$, which will play an essential role in the main proof.

Definition 2.2. The *Hilbert scheme of ordered points*, denoted $X^{\llbracket n \rrbracket}$, is the scheme defined by the Cartesian diagram

$$\begin{array}{ccc} X^{\llbracket n \rrbracket} & \longrightarrow & X^{[n]} \\ \downarrow & & \downarrow \\ X^n & \longrightarrow & \text{Sym}^n(X), \end{array}$$

where the right hand arrow is the Hilbert-Chow morphism taking a subscheme Z to its support cycle. Denote by $(Z, (x_i))$ the point in $X^{\llbracket n \rrbracket}$ mapping to $Z \in X^{[n]}$ and $(x_i) \in X^n$.

Roughly speaking, the advantage of introducing $X^{\llbracket n \rrbracket}$ is that it has a natural map to X^n . This makes it easier to handle than $X^{[n]}$, which maps naturally to the more complicated $\text{Sym}^n(X)$. Also, as $X^{\llbracket n \rrbracket} \rightarrow X^{[n]}$ is finite, we can reduce questions about homology classes on $X^{[n]}$ to questions about similar classes on $X^{\llbracket n \rrbracket}$.

We now give the definition of geometric subsets of $X^{[n]}$ and of $X^{\llbracket n \rrbracket}$, along with some results on these which will be needed later.

Let $\text{Hilb}_0^n(\mathbb{C}^d)$ be the punctual Hilbert scheme, defined as the closed subset of $\text{Hilb}^n(\mathbb{C}^d)$ parametrizing subschemes supported at the origin. We define punctual geometric subsets to be the constructible subsets of the punctual Hilbert scheme containing all 0-dimensional schemes of given isomorphism types.

Definition 2.3. A punctual geometric subset $Q \subseteq \text{Hilb}_0^n(\mathbb{C}^d)$ is a constructible set such that: If $Z \in Q$ and $Z' \in \text{Hilb}_0^n(\mathbb{C}^d)$ are such that $Z \cong Z'$ (as abstract schemes), then $Z' \in Q$.

A collection of punctual geometric subsets will naturally define subsets of both $X^{[n]}$ and $X^{\llbracket n \rrbracket}$.

Definition 2.4. Let Q_1, \dots, Q_k be punctual geometric subsets such that $Q_i \subseteq \text{Hilb}_0^{n_i}(\mathbb{C}^d)$. Let $A = (A_1, \dots, A_k)$ be a k -tuple of subsets of $\{1, \dots, n\}$, such that $|A_i| = n_i$, and such that the A_i define a partition of $\{1, \dots, n\}$.

- Define $P(Q_1, \dots, Q_k) \subseteq X^{\llbracket n \rrbracket}$ as the set of all $Z = Z_1 \sqcup \dots \sqcup Z_k$, where every Z_i is isomorphic to a $Z'_i \in Q_i$.
- Define $T(Q_1, \dots, Q_k, A) \subset X^{\llbracket n \rrbracket}$ as the set of all $(Z, (x_i))$ such that $Z = Z_1 \sqcup \dots \sqcup Z_k$, where every Z_i is isomorphic to a $Z'_i \in Q_i$, and such that $x_i = \text{Supp } Z_j$ if $i \in A_j$.

We now give the definition of geometric subsets of $X^{[n]}$ and $X^{\llbracket n \rrbracket}$.

Definition 2.5.

- A subset $P \subseteq X^{[n]}$ is geometric if it can be expressed using sets of the form $P(Q_1, \dots, Q_k)$ and a finite composition of the operations union, intersection and complement.
- A subset $T \subseteq X^{\llbracket n \rrbracket}$ is geometric if it can be expressed using sets of the form $T(Q_1, \dots, Q_k, A)$ and a finite composition of the operations of union, intersection and complement.

An equivalent definition which will be convenient in the proof is the following.

Definition 2.6.

- A subset $P \subseteq X^{[n]}$ is geometric if it can be expressed using sets of the form $\overline{P(Q_1, \dots, Q_k)}$, where the Q_i are closed and irreducible, together with a finite composition of the operations of union, intersection and complement.
- A subset $T \subseteq X^{\llbracket n \rrbracket}$ is geometric if it can be expressed using sets of the form $\overline{T(Q_1, \dots, Q_k, A)}$, where the Q_i are closed and irreducible, together with a finite composition of the operations of union, intersection and complement.

The equivalence of Definitions 2.5 and 2.6 is shown in Lemma 2.7 (vii).

Example. If $n = 1$ the only geometric subsets of $X^{[1]} = X$ are \emptyset and X . If $n = 2$, there are three geometric subsets: The sets \emptyset , $X^{[2]}$ and the subset parametrizing length 2 subschemes with support in one point. When X is a surface, a naturally occurring example of a geometric subset is the set $W \subset X^{[3\delta]}$, defined as the closure of

$$\{Z \in X^{[3\delta]} \mid Z = Z_1 \sqcup \dots \sqcup Z_\delta, \ Z_i = \text{Spec } \mathcal{O}_{S, x_i} / \mathfrak{m}_{x_i}^2\}.$$

This is the set that appears in Tzeng's proof of the Göttsche Conjecture.

It is easy to check that a geometric subset P is constructible.

Clearly, if $P \subseteq X^{[n]}$ is geometric and $Z \in P$, then for any $Z' \in X^{[n]}$ such that $Z \cong Z'$ we have $Z' \in P$. In other words, a geometric subset P is a union of isomorphism classes of subschemes $Z \in X^{[n]}$.

We may define the notion of isomorphism between points of $X^{\llbracket n \rrbracket}$ by saying that $(Z, (x_i)) \cong (Z', (x'_i))$ if there exists an isomorphism $Z \cong Z'$ which takes $Z|_{x_i}$ to $Z'|_{x'_i}$ for every i . Then similarly a geometric subset $T \subseteq X^{\llbracket n \rrbracket}$ is a union of isomorphism classes of pairs.

Let X and Y be d -dimensional, smooth, projective varieties, and let P and Q be geometric subsets of $X^{[n]}$ and $Y^{[n]}$ respectively. We say that P and Q are of the same *type* if the isomorphism classes of the points in P are the same as the isomorphism classes of points in Q . Clearly, for any geometric subset of X there is a unique geometric subset of Y of the same type. The type of a geometric subset $T \subseteq X^{\llbracket n \rrbracket}$ is defined in the same way.

Let $p : X^{\llbracket n \rrbracket} \rightarrow X^{[n]}$ be the natural morphism. The following lemma contains the essential facts about geometric subsets.

Lemma 2.7. *Let $P \subseteq X^{[n]}$ and $T \subseteq X^{\llbracket n \rrbracket}$ as sets.*

- (i) P is geometric $\Leftrightarrow p^{-1}(P)$ is geometric.
- (ii) T is geometric $\Rightarrow p(T)$ is geometric.
- (iii) P is geometric $\Leftrightarrow P$ is a finite union of sets of the form $P(Q_i)$.
- (iv) T is geometric $\Leftrightarrow T$ is a finite union of sets of the form $\overline{T((Q_i), A)}$.
- (v) P is geometric, closed and irreducible $\Leftrightarrow P$ is of the form $\overline{P((Q_i)}$ for closed, irreducible Q_i .
- (vi) T is geometric, closed and irreducible $\Leftrightarrow T$ is of the form $\overline{T((Q_i), A)}$ for closed, irreducible Q_i .

(vii) *Definitions 2.5 and 2.6 are equivalent.*

Proof. In this proof, we use the term geometric subsets for the sets satisfying Definition 2.5.

(iv) It suffices to show that intersections and complements of sets of the form $T((Q_i), A)$ are expressible as unions of such sets. Let $T((Q_i), A)$ and $T((Q'_i), A')$ be sets such that we have $A = (A_1, \dots, A_k)$ and $A' = (A'_1, \dots, A'_l)$. Then, if $T((Q_i), A) \cap T((Q'_i), A') \neq \emptyset$, we have $l = k$ and the k -tuple A is a permutation of the k -tuple A' . In this case, we may relabel the indices of the A'_i to get $A = A'$, and then $T((Q_i), A) \cap T((Q'_i), A) = T((Q_i \cap Q'_i), A)$.

Next we see that for any $T((Q_i), A)$ the set $X^{[n]} \setminus T((Q_i), A)$ is the union of all sets $T((\text{Hilb}_0^{n_i}(\mathbb{C}^d)), A')$ where A' is not a permutation of A , and the k sets of the form

$$T(\text{Hilb}_0^{n_1}(\mathbb{C}^d), \dots, \text{Hilb}_0^{n_i}(\mathbb{C}^d) \setminus Q_i, \dots, \text{Hilb}_0^{n_k}(\mathbb{C}^d), A)$$

for $i = 1, \dots, k$.

(ii) Follows from (iv) and $p(T((Q_i), A)) = P((Q_i))$.

(i) \Rightarrow follows from the fact that $p^{-1}(P(Q_i))$ is the union of $T((Q_i), A)$ for all admissible A . \Leftarrow then follows from (ii) and surjectivity of p .

(iii) Follows from (i), (ii), (iv) and the surjectivity of p .

(v) By (iii), we may write $P = \cup_j P((Q_{i,j})_i)$, and as P is closed, we have $P = \overline{\cup_j P((Q_{i,j})_i)} = \cup_j \overline{P((Q_{i,j})_i)} = \cup_j P(\overline{(Q_{i,j})_i})$. Irreducibility of P implies $P = \overline{P((Q_{i,j})_i)}$ for some j , so we may take $Q_i = \overline{Q_{i,j}}$. It remains to show that the Q_i can be chosen to be irreducible. Suppose not, then we have for instance Q_1 reducible. Let $Q_1 = \cup_j Q_{1,j}$ be the decomposition of Q_1 into closed, irreducible subsets. Each $Q_{1,j}$ must be equal to the closure of its orbit under the natural action of $\text{Aut}(\mathcal{O}_{\mathbb{A}^d, 0}/\mathfrak{m}_0^{n_1})$ on $\text{Hilb}_0^{n_1}(\mathbb{C}^d)$, hence we see that the $Q_{1,j}$ are geometric.

We then have $P = \cup_j P(Q_{1,j}, Q_2, \dots, Q_k)$, and as P is irreducible we may replace Q_1 with some $Q_{1,j}$. Repeat to get all Q_i irreducible, proving the \Rightarrow implication. The \Leftarrow implication is easy and omitted, but note that it depends on the hypothesis that X is connected.

(vi) Similar to (v).

(vii) It is obvious that a P satisfying Def. 2.6 satisfies 2.5. For the converse, note that the closed geometric P generate all geometric subsets by unions, intersections and complements. The proof of (v) shows that a closed geometric P is the union of sets of the form $\overline{P((Q_i))}$ with Q_i closed and irreducible. Hence closed, geometric P satisfy Definition 2.6, and the claim follows. The case of T is similar. \square

3. OUTLINE OF PROOF

We give an outline of the proof of the main theorem. For ease of presentation, we restrict our attention in the outline to Theorem 1.1 (i), ignoring the extra complications of (ii) and Theorem 1.2. Here and in the formal proof, we assume that the number of vector bundles E_i is 1; the general case is no more difficult. By Lemma 2.7 we see that the irreducible components of P are geometric of type depending only on the type of P , hence we may assume that P is irreducible.

The set-up is then that we are given a closed, irreducible geometric subset P of $X^{[n]}$, a (1-variable) Chern polynomial F and a vector bundle E , and we want to show that

$$\deg F(E^{[n]}) \cap [P]$$

is given by a universal polynomial.

3.1. Reduction to $X^{[n]}$. The first step is to replace $X^{[n]}$ with the Hilbert scheme of ordered points $X^{[n]}$ of Def. 2.2. Define the bundle $E^{[n]}$ on $X^{[n]}$ to be the pullback of $E^{[n]}$ along $X^{[n]} \rightarrow X^{[n]}$.

Lemma 4.2 gives a closed, irreducible $T \subseteq X^{[n]}$ which is geometric, maps properly and finitely onto P , and which is such that $\deg(T/P)$ and the type of T are determined by the type of P . The projection formula shows that

$$\deg(T/P) \left(\deg F \left(E^{[n]} \right) \cap [P] \right) = \deg F \left(E^{[n]} \right) \cap [T].$$

Thus, it suffices to show that $\deg F \left(E^{[n]} \right) \cap [T]$ is given by a universal polynomial.

3.2. Approximating Constructions. Next, following [Li], we let α be a partition of $\{1, \dots, n\}$, and define the scheme $X^{[\alpha]}$. Considering α as a set of subsets of $\{1, \dots, n\}$, we let

$$X^{[\alpha]} := \prod_{A \in \alpha} X^{[|A|]}.$$

So, for example, if α is the partition of $\{1, \dots, n\}$ into n one-element sets, we have $X^{[\alpha]} = X^n$. At the other extreme, let Λ denote the trivial partition of $\{1, \dots, n\}$ into one set, we then have $X^{[\Lambda]} = X^{[n]}$. Informally, the scheme $X^{[\alpha]}$ parametrizes ordered collections of n points in X , with the additional data that when k points whose labels are in the same set in the partition α come together at x , one must specify a length k subscheme supported at x .

For every α , there is a natural rational map $f : X^{[n]} \dashrightarrow X^{[\alpha]}$, defined on the open set where the moduli problem the two schemes solve is the same. We define a bundle $E^{[\alpha]}$ such that $f^*(E^{[\alpha]}) = E^{[n]}$ on the locus where f is defined. If α is such that T intersects the locus where f is defined, we define a closed, irreducible subset $T_\alpha = \overline{f(T)} \subseteq X^{[\alpha]}$. We have $E^{[\Lambda]} = E^{[n]}$ and $T_\Lambda = T$.

All the T_α are pairwise birational in a natural way, which induces a graph like set $\Gamma \subset \prod_\alpha T_\alpha$, where the product is over all α such that T_α is defined. Define the scheme Q as the closure of Γ . The projections induce proper, birational morphisms from Q to every T_α .

We define the class $C_\alpha \in H^*(Q)$ by

$$C_\alpha = F \left(E^{[\alpha]} \right),$$

suppressing the pullback of $E^{[\alpha]}$ along $Q \rightarrow T_\alpha$. Let $C = C_\Lambda$. By the projection formula, $\deg C \cap [Q] = \deg F \left(E^{[n]} \right) \cap [T]$, so the proof of the main theorem reduces to showing that $\deg C \cap [Q]$ is universal.

We next define a class $D = C + \sum_{\alpha \neq \Lambda} k_\alpha C_\alpha$, where the k_α are certain combinatorially defined integers. There is a natural morphism $Q \rightarrow X^n$, and one should think of the class D as being supported on (a neighbourhood of) the set $Q|_\Delta$, where $\Delta \subset X^n$ is the small diagonal. The choice of the integers k_α is motivated by the fact that they force D to vanish on the complement of this locus.

For any $\alpha \neq \Lambda$, the scheme $X^{[\alpha]}$ is by definition a product of schemes $X^{[m]}$ with $m < n$. This induces product decompositions of $E^{[\alpha]}$ and T_α , which allow us to express $\deg C_\alpha \cap [Q]$ in terms of integrals of Chern classes of $E^{[m]}$ over geometric subsets of $X^{[m]}$ with $m < n$. By induction on n we can thus show that $\deg C_\alpha \cap [T_\alpha]$ is universal for $\alpha \neq \Lambda$. This argument gives Lemma 4.13, by which it suffices to show that

$$\deg D \cap [Q]$$

is universal.

3.3. Relative Constructions. Consider the tangent bundle $TX \rightarrow X$, and let $\overline{TX} := \mathbb{P}(\mathcal{O}_X \oplus TX)$ be the natural compactification. Let $\text{Hilb}^n(\overline{TX}/X)$ be the relative Hilbert scheme. Emulating the definition of Q and $E^{[\alpha]}$ with $\text{Hilb}^n(\overline{TX}/X)$ replacing $X^{[n]}$, we define the scheme \mathcal{Q} and the bundles $\mathcal{E}^{[\alpha]}$ on \mathcal{Q} . The classes $\mathcal{C}_\alpha, \mathcal{D} \in H^*(\mathcal{Q})$ are defined similarly to C_α and D .

Denote by \overline{TX}^n the n -fold fibre product of \overline{TX} over X . There are natural morphisms $g : Q \rightarrow X^n$ and $h : \mathcal{Q} \rightarrow \overline{TX}^n$, where g is given by composing $Q \rightarrow X^{[n]}$ and $X^{[n]} \rightarrow X^n$, and h is defined similarly. Let $\Delta \subset X^n$ be the small diagonal, and consider X as a subset of \overline{TX}^n using n copies of the 0-section.

Let $U \subset Q$ and $\mathcal{U} \subset \mathcal{Q}$ be Euclidean open neighbourhoods of $g^{-1}(\Delta)$ and $h^{-1}(X)$, respectively. Choosing U and \mathcal{U} small enough, we can define a topological isomorphism $f : U \rightarrow \mathcal{U}$ inducing (topological) isomorphisms of the bundles $f^*(\mathcal{E}^{[\alpha]}) \cong E^{[\alpha]}$.

The map f is defined starting from (the inverse of) an exponential map on X , analytic on the fibres of TX , which gives a continuous map $X^{[n]} \rightarrow \text{Hilb}^n(\overline{TX}/X)$, defined in a neighbourhood of the locus of subschemes supported at a single point. By the similarity in the definition of Q and \mathcal{Q} we show that this induces $f : Q \rightarrow \mathcal{Q}$. The details of this construction are contained in Lemmas 5.1-5.3.

Recall that the class D was defined as a sum of C and a linear combination of the C_α . Now, restricting the classes of this sum to $Q \setminus U$ it can be shown the terms of this sum will cancel, as a consequence of the fact that there are canonical local isomorphisms between the $E^{[\alpha]}$. Hence, D vanishes when restricted to $Q \setminus U$. Similarly, \mathcal{D} vanishes upon restriction to $\mathcal{Q} \setminus \mathcal{U}$.

Thus, informally, the classes D and \mathcal{D} are “concentrated” over U and \mathcal{U} , respectively. Making this more precise, we note that there are relative cohomology classes $\overline{D} \in H^*(Q, Q \setminus U)$ and $\overline{\mathcal{D}} \in H^*(\mathcal{Q}, \mathcal{Q} \setminus \mathcal{U})$ lifting D and \mathcal{D} .

There is a map $f^* : H^*(\mathcal{Q}, \mathcal{Q} \setminus \mathcal{U}) \rightarrow H^*(Q, Q \setminus U)$, defined by excision after shrinking U and \mathcal{U} . Lemma 4.19 shows that we can choose \overline{D} and $\overline{\mathcal{D}}$ in such a way that the map $f^*(\overline{\mathcal{D}}) = \overline{D}$. As a consequence of this, we find

$$\deg D \cap [Q] = \deg \mathcal{D} \cap [\mathcal{Q}].$$

The proof of Lemma 4.19 is rather technical and occupies Section 5.

3.4. Pullback from the Grassmannian. It remains to show that $\deg \mathcal{D} \cap [\mathcal{Q}]$ is universal. To this end, note first that \mathcal{Q} is defined by the data of the type of T and the tangent bundle $TX \rightarrow X$, which roughly implies that it can be pulled back from a universal construction over a Grassmannian.

Let $U \rightarrow \text{Gr}(d, N)$ be the universal rank d subbundle over a Grassmannian. Here N is any integer large enough that TX embeds as a topological subbundle of \mathcal{O}_X^N , so that we have a continuous classifying map $f : X \rightarrow \text{Gr}(d, N)$ with $TX \cong f^*(U)$ as topological bundles. We may define the scheme \mathcal{Q}_{Gr} by the same construction as \mathcal{Q} , replacing $\text{Hilb}^n(\overline{TX}/X)$ with $\text{Hilb}^n(\overline{U}/\text{Gr}(d, N))$. There is a natural morphism $\mathcal{Q}_{\text{Gr}} \rightarrow \text{Gr}(d, N)$ and a Cartesian diagram of topological spaces.

$$\begin{array}{ccc} \mathcal{Q} & \longrightarrow & \mathcal{Q}_{\text{Gr}} \\ p \downarrow & & \downarrow \\ X & \xrightarrow{f} & \text{Gr}(d, N). \end{array}$$

Let $h : X \rightarrow \mathrm{Gr}(r, M)$ be a classifying map for E as a topological bundle, and consider the Cartesian diagram

$$\begin{array}{ccc} \mathcal{Q} & \xrightarrow{g} & \mathcal{Q}_{\mathrm{Gr}} \times \mathrm{Gr}(r, M) \\ p \downarrow & & \downarrow \\ X & \xrightarrow{(f, h)} & \mathrm{Gr}(d, N) \times \mathrm{Gr}(r, M). \end{array}$$

Note that the horizontal arrows in these diagrams are *not* required to be analytic, and we are here dealing with the underlying topological spaces. We define bundles $\mathcal{E}_{\mathrm{Gr}}^{[\alpha]}$ on $\mathcal{Q}_{\mathrm{Gr}} \times \mathrm{Gr}(r, M)$ such that $g^*(\mathcal{E}_{\mathrm{Gr}}^{[\alpha]}) = \mathcal{E}^{[\alpha]}$.

Using this we then show that there is a class $\mathcal{G} \in H^*(\mathrm{Gr}(d, N) \times \mathrm{Gr}(r, M))$, depending only on F and the type of T , such that

$$p_*(\mathcal{D} \cap [\mathcal{Q}]) = (f, h)^*(\mathcal{G}) \cap [X].$$

Now, as the rational cohomology ring of a Grassmannian is generated by the Chern classes of its universal bundle, $(f, h)^*(\mathcal{G})$ is a polynomial in the Chern classes of T_X and E . We show that this polynomial is independent of the choice of N and M . Hence $\deg \mathcal{D} \cap [\mathcal{Q}]$ is a universal linear combination of the Chern numbers of (X, E) , which concludes the proof of the main theorem.

4. PROOF OF MAIN THEOREM

We begin the formal proof of the main theorem. For notational simplicity we assume that $k = 1$, so there is only one vector bundle E involved; the general case is essentially the same. To avoid dealing with Theorem 1.1 and 1.2 separately, we use the following convention: When a formula includes $T_{X^{[n]}}$, terms involving $T_{X^{[n]}}$ should be ignored unless $\dim X = 1, 2$, as should any statement involving $T_{X^{[n]}}$.

For part (i), it suffices to show treat the case where P is irreducible. If P is closed and irreducible, we have $c_{\mathrm{SM}}(P) = [P] +$ terms of lower homological degree, hence it suffices to show (ii). By the inclusion-exclusion property of CSM classes, it finally suffices to prove (ii) for closed and irreducible P .

In this section and the next, we have X, P, E, d and F as in the main theorem, assuming additionally that P is closed and irreducible.

Note. Starting with $X^{[n]}$, we will define several schemes equipped with morphisms to X^n . Given such an $f : Y \rightarrow X^n$ and a subset U of X^n , we write “the restriction of Y to U ” to mean $f^{-1}(U)$, and denote this scheme by $Y|_U$.

4.1. Reduction to $X^{[n]}$. Recall the definition of the Hilbert scheme of ordered points $X^{[n]}$, given by the Cartesian diagram

$$\begin{array}{ccc} X^{[n]} & \longrightarrow & X^{[n]} \\ \downarrow & & \downarrow \\ X^n & \longrightarrow & \mathrm{Sym}^n(X), \end{array}$$

where the right hand arrow is the Hilbert-Chow morphism.

Definition 4.1. Denote the pullbacks of $E^{[n]}$ and $T_{X^{[n]}}$ along $X^{[n]} \rightarrow X^{[n]}$ by $E^{[n]}$ and $T_{X^{[n]}}^{[n]}$, respectively.

We will use the projection formula to relate the degree of $F(E^{[n]}, T_{X^{[n]}}) \cap [P]$ to a similar class involving $E^{[n]}$ and $T_{X^{[n]}}^{[n]}$ on $X^{[n]}$. The first step is to produce a closed, irreducible geometric subset $T \subseteq X^{[n]}$ mapping properly onto P with universal degree.

Lemma 4.2. *There exists a closed, irreducible geometric $T \subset X^{\llbracket n \rrbracket}$, such that $X^{\llbracket n \rrbracket} \rightarrow X^{[n]}$ maps T finitely onto P . Up to a permutation of $\{1, \dots, n\}$ inducing automorphisms of X^n and hence of $X^{\llbracket n \rrbracket}$, the type of this T depends only on the type of P .*

Proof. As P is closed and irreducible, by Lemma 2.7 (v) we have $P = \overline{P((Q_i))}$ for closed and irreducible punctual geometric subsets $Q_i \subset \text{Hilb}_0^{n_i}(\mathbb{C}^d)$. We take $A = (A_1, \dots, A_k)$ to be a k -element partition of $\{1, \dots, n\}$ such that $|A_k| = n_k$, and let $T = \overline{T((Q_i), A)}$. The claims of the lemma are easy to check. \square

Recall that $c_M(T)$ denotes the Chern-Mather class of T .

Lemma 4.3. *Let T be a closed, irreducible geometric subset of $X^{\llbracket n \rrbracket}$. In order to prove the main theorem, it suffices to show that*

$$\deg F \left(E^{\llbracket n \rrbracket}, T_{X^{\llbracket n \rrbracket}}^{\llbracket n \rrbracket} \right) \cap c_M(T)$$

is given by a universal polynomial depending only on F and the type of T , such that the degree of the polynomial is at most l , where l is the maximum number of components of Z for $(Z, (x_i)) \in T$.

Proof. Let $p : X^{\llbracket n \rrbracket} \rightarrow X^{[n]}$ be the natural morphism. By the inclusion-exclusion property of CSM classes, it suffices to show the main theorem when $P \subseteq X^{[n]}$ is closed and irreducible. Let $T \subseteq X^{\llbracket n \rrbracket}$ be a closed, irreducible geometric set mapping finitely onto P , as provided by Lemma 4.2.

The level sets of the local Euler obstruction ν_T are geometric subsets contained in T . This can be shown as follows. By Lemma 2.7 (iv), we may write T as the union of sets of the form $T((Q_i), A)$. Let $A = (A_1, \dots, A_k)$, and suppose $(Z, (x_i)) \in T((Q_i), A) \subseteq T$. We have $Z = \sqcup_i Z_i$ with Z_i of isomorphism type in Q_i and $x_j = \text{Supp } Z_i$ for $j \in A_i$. The local analytic type of $(Z, (x_i))$ in T is determined by the isomorphism types of the Z_i . Furthermore, there exists a local analytic neighbourhood of $(Z, (x_i))$ in T which decomposes as a product $\prod_{i=1}^k U_i$, where the analytic type of U_i is determined by the isomorphism type of Z_i .

We then have

$$\nu_T(Z, (x_i)) = \nu_{\prod U_i}(Z, (x_i)) = \prod_i \nu_{U_i}(Z_i).$$

The factor $\nu_{U_i}(Z_i)$ depends only on the isomorphism type of Z_i . This implies that the level sets of ν_T intersected with $T((Q_i), A)$ are geometric. It follows that the complete level sets of ν_T are geometric.

As we have $c_{\text{SM}}(\nu_T) = c_M(T)$ by definition, the class $c_{\text{SM}}(T)$ satisfies

$$c_{\text{SM}}(T) = c_{\text{SM}}(1_T) = c_{\text{SM}}(\nu_T) + c_{\text{SM}}(1_T - \nu_T) = c_M(T) + \sum ic_{\text{SM}}(T_i)$$

where the sum is finite and the T_i are geometric subsets of $X^{\llbracket n \rrbracket}$ of lower dimension than T , with type depending only on the type of T . By induction on $\dim T$ the hypothesis of the lemma implies that $\deg F(E^{\llbracket n \rrbracket}, T_{X^{\llbracket n \rrbracket}}^{\llbracket n \rrbracket}) \cap c_{\text{SM}}(T)$ is universal. The functorial property of CSM classes implies

$$p_*(c_{\text{SM}}(T)) = \deg(T/P)c_{\text{SM}}(P) + \sum ic_{\text{SM}}(P_i),$$

where the P_i are subsets of $X^{[n]}$, of lower dimension than P . It is not hard to show that the P_i are geometric of type depending only on the type of P . Induction on the dimension of P now gives the main theorem as claimed. \square

4.2. Partitions. We will define schemes $X^{[\alpha]}$ approximating $X^{[n]}$, where α is a partition of n . First, we fix notation and conventions with respect to partitions.

Definition 4.4. A *partition of n* is a set α of subsets of $\{1, \dots, n\}$, such that distinct elements of α are disjoint, and such that the union of all elements in α equals $\{1, \dots, n\}$.

Note that this conflicts with a common usage of the term, where “partition of n ” means a way of writing n as a sum of positive integers.

The following definition summarizes the relevant notation for partitions.

Definition 4.5. If α is a partition of n , let \sim_α be the equivalence relation on $\{1, \dots, n\}$ given by letting the elements of α form equivalence classes.

Define a partial ordering on the set of all partitions of n by letting $\alpha \leq \beta$ if every element of α is contained in an element of β . Equivalently, we say $\alpha \leq \beta$ if \sim_α is a finer relation than \sim_β .

Denote by Λ the maximal partition under this ordering, that is, $\Lambda = \{\{1, \dots, n\}\}$.

Given two partitions α, β , we denote by $[\alpha, \beta]$ the set of partitions γ such that $\alpha \leq \gamma \leq \beta$, and define $[\alpha, \beta)$ etc. similarly.

4.3. Approximating Constructions. From this point on, we fix a closed, irreducible, geometric subscheme $T \subseteq X^{[n]}$. In this section, we define the schemes $X^{[\alpha]}$, the bundles $E^{[\alpha]}, T_{X^{[n]}}^{[\alpha]}$, the subsets $T_\alpha \subseteq X^{[\alpha]}$, and the approximate classes C_α and D .

Definition 4.6. If α is a partition of n , define the scheme $X^{[\alpha]}$ by

$$X^{[\alpha]} = \prod_{A \in \alpha} X^{[A]},$$

where $X^{[A]} \cong X^{[|A|]}$ and parametrizes pairs $(Z, (p_i)_{i \in A})$ such that $\sum_{i \in A} p_i$ is the fundamental cycle of Z .

There is a natural morphism $X^{[\alpha]} \rightarrow X^n$ defined by the decomposition $X^n = \prod_{A \in \alpha} X^A$ and the natural morphisms $X^{[A]} \rightarrow X^A$.

Definition 4.7. Define the vector bundles $E^{[\alpha]}$ and $T_{X^{[n]}}^{[\alpha]}$ on $X^{[\alpha]}$ by

$$E^{[\alpha]} = \bigoplus_{A \in \alpha} E^{[A]} \quad T_{X^{[n]}}^{[\alpha]} = \bigoplus_{A \in \alpha} T_{X^{[A]}}^{[A]}$$

where we suppress pullback along the projection $X^{[\alpha]} \rightarrow X^{[A]}$.

For suitable α we will define schemes $T_\alpha \subseteq X^{[\alpha]}$ birational to T .

Definition 4.8. If α is a partition of n , denote by Δ_α the subset of X^n given by

$$\Delta_\alpha = \{(x_1, \dots, x_n) \in X^n \mid x_i = x_j \text{ if } i \sim_\alpha j\}.$$

We refer to the sets Δ_α as *diagonals*.

Writing $T = \overline{T((Q_i), A)}$ as in Lemma 2.7 (vi), it is easy to see that the image of T in X^n is a diagonal.

Definition 4.9. Let μ be the partition such that $\Delta_\mu \subseteq X^n$ is the image of T under $X^{[n]} \rightarrow X^n$.

Note that if $T = \overline{T((Q_i), A)}$ and $A = (A_1, \dots, A_k)$, we have $\mu = \{A_1, \dots, A_k\}$.

Let α be a partition, and let $f_\alpha : X^{[n]} \dashrightarrow X^{[\alpha]}$ be the natural local isomorphism, defined on the open set where the moduli problems $X^{[n]}$ and $X^{[\alpha]}$ solve are the same. Specifically, f_α is defined on the set of points $(Z, (x_i))$ where $x_i \neq x_j$ for all i, j with $i \not\sim_\alpha j$.

It is easy to check that the locus where f_α is naturally defined intersects T if and only if $\alpha \geq \mu$. If this is the case, we let T_α be the closure of $f_\alpha(T)$ in $X^{[\alpha]}$.

Recall that $\widetilde{T}_\alpha \rightarrow T_\alpha$ denotes the Nash blow up. As $\widetilde{T}_\alpha \rightarrow T_\alpha$ is birational, there is a natural rational map $g_\alpha : T \dashrightarrow \widetilde{T}_\alpha$.

Definition 4.10. Let

$$g := (g_\alpha)_{\alpha \geq \mu} : T \dashrightarrow \prod_{\alpha \geq \mu} \widetilde{T}_\alpha,$$

and define Q to be the closure of $g(T)$ in $\prod \widetilde{T}_\gamma$.

For every α there are birational proper morphisms $Q \rightarrow \widetilde{T}_\alpha \rightarrow T_\alpha$. Any cohomology class on T_α and \widetilde{T}_α may be pulled back along these morphisms without changing the degree. As our goal is to compute the degrees of such classes, we will suppress pullbacks of bundles and cohomology classes.

4.4. Approximations of the Cohomology Classes. The schemes and bundles indexed by partitions α give rise to “approximate” cohomology classes, which we now define. Recall that Λ denotes the maximal partition of n , and that $T_{\widetilde{T}_\alpha}$ is the Nash bundle on \widetilde{T}_α .

Definition 4.11. Let α be a partition $\geq \mu$. Define the class $C_\alpha \in H^*(Q)$ by

$$C_\alpha = F\left(E^{[\alpha]}, T_{X^{[n]}}^{[\alpha]}\right) \cup c_\bullet\left(T_{\widetilde{T}_\alpha}\right).$$

We let $C = C_\Lambda$.

Note that the main theorem is reduced to the claim that $\deg(C \cap [Q])$ is universal.

Definition 4.12. Let α be a partition $\geq \mu$. Define the class $D_\alpha \in H^*(Q)$ by putting $D_\mu = C_\mu$, and for $\alpha > \mu$ let D_α be defined inductively by

$$D_\alpha = C_\alpha - \sum_{\gamma \in [\mu, \alpha)} D_\gamma.$$

We let $D = D_\Lambda$.

Remark. It is not hard to show that in fact $D = \sum_{\alpha \geq \mu} (-1)^{|\alpha|-1} (|\alpha| - 1)! C_\alpha$. Except in the proof of Proposition 6.2, we will not need this, and we work instead directly with the inductive definition of D .

The motivation behind the definition of D_α is the following. Firstly, it follows trivially from the definition that if $\deg(D \cap [Q])$ and $\deg(C_\alpha \cap [Q])$ are universal for $\alpha \neq \Lambda$, then $\deg(C \cap [Q])$ is universal as well. Secondly, D is chosen such that the restriction of D to $Q \setminus (Q|_{\Delta_\Lambda})$ vanishes, which allows us to reduce the computation of its degree to studying a small neighbourhood of $Q|_{\Delta_\Lambda}$.

4.5. Reduction to $\deg(D \cap [Q])$. If α is a nonmaximal partition of n , the scheme $X^{[\alpha]}$ is by definition a product of schemes $X^{[m]}$ with $m < n$. As a consequence of this, the computation of C_α can be reduced to similar computations on such $X^{[m]}$. This yields the following induction result.

Lemma 4.13. *Let m be a positive integer. Suppose that Theorem 1.1/1.2 holds for every $n < m$, and suppose that for $n = m$ the degree of $D \cap [Q]$ is given by a universal linear polynomial in the Chern numbers of (X, E) . Then Theorem 1.1/1.2 holds for $n = m$.*

Proof. Assume that the theorem holds for every $n < m$. We shall then show that for every partition $\alpha \in [\mu, \Lambda)$, the degree of C_α is expressed by a universal polynomial. We have $C = D_\Lambda + \sum_{\alpha \in [\mu, \Lambda)} k_\alpha C_\alpha$, hence the statement of the lemma follows from this.

Let α be a partition $\geq \mu$. Recall first that by definition there is a product decomposition

$$X^{[\alpha]} = \prod_{A \in \alpha} X^{[A]}.$$

This gives rise to a product decomposition $T_\alpha = \prod_{A \in \alpha} T_A$, where the $T_A \subseteq X^{[A]}$ are closed, irreducible, geometric subsets. Since the Nash blow up preserves products, we have $\widetilde{T}_\alpha = \prod \widetilde{T}_A$, as well as the bundle decompositions $E^{[\alpha]} = \oplus E^{[A]}$, $T_{\widetilde{T}_\alpha} = \oplus T_{\widetilde{T}_A}$ and $T_{X^{[n]}}^{[\alpha]} = \oplus T_{X^{[n]}}^{[A]}$.

Now, using the Whitney sum formula we can find a universal expression for

$$C_\alpha = F\left(E^{[\alpha]}, T_{X^{[n]}}^{[\alpha]}\right) \cdot c_\bullet\left(T_{\widetilde{T}_\alpha}\right)$$

as a polynomial in the Chern classes of $E^{[A]}$, $T_{X^{[n]}}^{[A]}$, and $T_{\widetilde{T}_A}$ for different $A \in \alpha$. Assuming that $\alpha < \Lambda$, we have $|A| < m$ for every $A \in \alpha$. By the induction hypothesis, we find a universal polynomial for

$$\deg C_\alpha \cap [Q] = \deg C_\alpha \cap [T_\alpha],$$

as required.

The claim about the degree of the universal polynomial G in the main theorem also follows by induction, using the assumption that $\deg(D \cap [Q])$ is linear as a polynomial in the Chern numbers of (X, E) . \square

Since the theorem is clear for $n = 1$, in order to complete the proof of it now suffices to show that the degree of $D \cap [Q]$ is given by a linear polynomial in the Chern numbers of (X, E) .

4.6. Relative Constructions. As previously mentioned, the class D vanishes when restricted to the part of Q lying over the complement of the small diagonal $\Delta_\Lambda \subset X^n$. It may thus essentially be computed by looking at a neighbourhood of $Q|_{\Delta_\Lambda}$. The next step is now to use this to show the degree of D equals that of a class $\mathcal{D} \in H^*(\mathcal{Q})$, where \mathcal{Q} is a scheme defined similarly to Q , but with $X^{[n]}$ replaced with the relative Hilbert scheme $\text{Hilb}^n(\overline{TX}/X)$.

We therefore repeat the constructions of approximating schemes and classes in this relative setting. These are for the most part straightforward analogues of the absolute constructions, except for the scheme \mathcal{T} that corresponds to T (and so every scheme derived from \mathcal{T} , i.e. $\mathcal{T}_\alpha, \mathcal{Q}$), where we impose the condition that the first marked point must lie in the 0-section $X \subset \overline{TX}$.

In order to integrate cohomology classes we need our schemes to be proper. Hence we let \overline{TX} denote the \mathbb{P}^d -bundle $\mathbb{P}(\mathcal{O}_X \oplus TX)$, with the convention that $\mathbb{P}(V)$ is the set of lines through the origin in V . Let $\pi : \overline{TX} \rightarrow X$ be the projection, and let \overline{TX}^n/X , $\text{Sym}^n(\overline{TX}/X)$ and $\text{Hilb}^n(\overline{TX}/X)$ denote the fibre product, relative symmetric product and relative Hilbert scheme, respectively.

Definition 4.14. Define the scheme $\overline{TX}^{[n]}$ by the cartesian diagram

$$\begin{array}{ccc} \overline{TX}^{[n]} & \longrightarrow & \text{Hilb}^n(\overline{TX}/X) \\ \downarrow & & \downarrow \\ \overline{TX}^n/X & \longrightarrow & \text{Sym}^n(\overline{TX}/X). \end{array}$$

Definition 4.15. Let $\mathcal{E}^{[n]}$ be the tautological bundle on $\text{Hilb}^n(\overline{TX}/X)$ corresponding to the vector bundle $\pi^*(E)$ on \overline{TX} , and let $\mathcal{E}^{[n]}$ be the pullback of $\mathcal{E}^{[n]}$ to $\overline{TX}^{[n]}$. If $\dim X = 1$ or 2 , let $T_{\overline{TX}^{[n]}}$ be the relative tangent bundle of $\text{Hilb}^n(\overline{TX}/X) \rightarrow X$, and denote its pullback to $\overline{TX}^{[n]}$ by $T_{\overline{TX}^{[n]}}^{[n]}$.

Definition 4.16. Let α be a partition of n . Define the scheme $\overline{TX}^{[\alpha]}$ by

$$\overline{TX}^{[\alpha]} = \prod_{A \in \alpha} \overline{TX}^{[A]},$$

where the product is fibre product over X . Define the bundles $\mathcal{E}^{[\alpha]}$ and $T_{\overline{TX}^{[n]}}^{[\alpha]}$ on $\overline{TX}^{[\alpha]}$ by

$$\mathcal{E}^{[\alpha]} = \bigoplus_{A \in \alpha} \mathcal{E}^{[A]}, \quad T_{\overline{TX}^{[n]}}^{[\alpha]} = \bigoplus_{A \in \alpha} T_{\overline{TX}^{[A]}}^{[A]}$$

suppressing notation for the natural pullbacks.

Let \mathcal{T}' be the subset of $\overline{TX}^{[n]}$ consisting of the labelled subschemes $(Z, (x_i))$ such that $(Z, (x_i))$ is isomorphic to a labelled subscheme in T , where isomorphism of labelled schemes is as defined above Lemma 2.7. Put differently, if $\overline{TX}_x^{[n]}$ denotes the fibre of $\overline{TX}^{[n]} \rightarrow X$ at x , then \mathcal{T}' is the set such that $\mathcal{T}' \cap \overline{TX}_x^{[n]}$ is a geometric subset of the same type as T for all $x \in X$.

Let $r : \overline{TX}^{[n]} \rightarrow \overline{TX}$ be the morphism defined by $r((Z, (x_i))) = x_1$, and define \mathcal{T} to be $r^{-1}(X) \cap \mathcal{T}'$, where $X \subset \overline{TX}$ is embedded by the 0-section.

For every partition α , there is a local isomorphism $f_\alpha : \overline{TX}^{[n]} \dashrightarrow \overline{TX}^{[\alpha]}$, defined where the moduli problems the two schemes solve are the same. Using these maps, we may replace T by \mathcal{T} in Definition 4.10 and the preceding paragraphs, thus defining the schemes \mathcal{T}_α (for $\alpha \geq \mu$), $\widetilde{\mathcal{T}}_\alpha$ and \mathcal{Q} . We omit the details.

Finally, we define the relative analogues of the classes C_α , D_α .

Definition 4.17. Let α be a partition $\geq \mu$. Define $\mathcal{C}_\alpha \in H^*(\mathcal{Q})$ by

$$\mathcal{C}_\alpha = F\left(\mathcal{E}^{[\alpha]}, T_{\overline{TX}^{[n]}}^{[\alpha]}\right) \cdot c_\bullet\left(T_{\widetilde{\mathcal{T}}_\alpha}\right).$$

Let $\mathcal{C} = \mathcal{C}_\Lambda$.

Definition 4.18. Let α be a partition $\geq \mu$. Define $\mathcal{D}_\alpha \in H^*(\mathcal{Q})$ by putting $\mathcal{D}_\mu = \mathcal{C}_\mu$, and for $\alpha > \mu$ define \mathcal{D}_α inductively by

$$\mathcal{D}_\alpha = \mathcal{C}_\alpha - \sum_{\gamma \in [\mu, \alpha)} \mathcal{D}_\gamma.$$

Let $\mathcal{D} = \mathcal{D}_\Lambda$.

4.7. Relating \mathcal{D}_Λ to \mathcal{D}_Λ . For notational convenience, we will in the following write \overline{TX}^n for the n -fold fibre product of \overline{TX} over X .

Let $Q_0 \subseteq Q$ be the restriction of Q to the small diagonal $\Delta_\Lambda \subset X^n$. Define $\mathcal{Q}_0 \subseteq \mathcal{Q}$ similarly as the restriction of \mathcal{Q} to the set $X \subset \overline{TX}^n$ where the inclusion of X is given by n copies of the 0-section. The classes \mathcal{D} and \mathcal{D} are related by the following lemma and its corollary.

Lemma 4.19. *There exists a pair of open neighbourhoods $U' \subset U$ of Q_0 in Q , a pair of open neighbourhoods $\mathcal{U}' \subset \mathcal{U}$ of \mathcal{Q}_0 in \mathcal{Q} , a homeomorphism $(U', U) \rightarrow (\mathcal{U}', \mathcal{U})$, and a class $\overline{\mathcal{D}} \in H^*(\mathcal{Q}, \mathcal{Q} \setminus \mathcal{U}')$ lifting $\mathcal{D} \in H^*(\mathcal{Q})$, such that the composition*

$$H^*(\mathcal{Q}, \mathcal{Q} \setminus \mathcal{U}') \rightarrow H^*(\mathcal{U}, \mathcal{U} \setminus \mathcal{U}') \rightarrow H^*(U, U \setminus U') \rightarrow H^*(Q, Q \setminus U') \rightarrow H^*(Q)$$

sends the class $\overline{\mathcal{D}}$ to \mathcal{D} .

Corollary 4.20. *The degree of $D \cap [Q]$ equals the degree of $\mathcal{D} \cap [Q]$.*

Proof of corollary. There are relative fundamental classes $[(Q, Q \setminus \mathcal{U}')] , [(\mathcal{U}, \mathcal{U} \setminus \mathcal{U}')] , [(U, U \setminus U')]$ and $[Q, Q \setminus U']$ in the appropriate homology groups. Replacing H^* with H_* in the above sequence (and reversing the arrows) each fundamental class is sent to the next, so in the composition the class $[Q]$ is sent to $[(Q, Q \setminus \mathcal{U})]$. This implies

$$\deg D \cap [Q] = \deg \overline{\mathcal{D}} \cap [(Q, Q \setminus \mathcal{U}')].$$

Now, $[(Q, Q \setminus \mathcal{U}')]$ is the push-forward of $[Q]$, which shows that $\deg \overline{\mathcal{D}} \cap [(Q, Q \setminus \mathcal{U}')] = \deg \mathcal{D} \cap [Q]$, completing the proof. \square

The proof of Lemma 4.19 is quite technical and is postponed to Section 5. We now show how the main theorem follows from Corollary 4.20.

Proof of main theorem. By Corollary 4.20, if $\deg \mathcal{D} \cap [Q]$ is given by a universal linear polynomial, the same is true for $\deg D \cap [Q]$, which by Lemma 4.13 would imply the main theorem.

Every construction made in Section 4.6 starting from $\overline{TX} \rightarrow X$ can be carried out with the bundle $TX \rightarrow X$ replaced by an arbitrary algebraic rank d vector bundle. In particular, we may construct the analogue of \mathcal{Q} starting from the universal rank d subbundle $U \rightarrow \text{Gr}(d, N)$, where N is any integer large enough that TX is the pullback of U along a continuous classifying map $X \rightarrow \text{Gr}(d, N)$. Call this scheme \mathcal{Q}_{Gr} , let $q : \mathcal{Q}_{\text{Gr}} \rightarrow \text{Gr}(d, N)$ be the natural morphism, and denote the analogues of \mathcal{T}_{α} by $\mathcal{T}_{\alpha, \text{Gr}}$.

Let r be the rank of E , and let M be a sufficiently large integer. There is then a Cartesian diagram

$$\begin{array}{ccc} \mathcal{Q} & \xrightarrow{f} & \mathcal{Q}_{\text{Gr}} \times \text{Gr}(r, M) \\ p \downarrow & & q \times \text{id} \downarrow \\ X & \xrightarrow{g} & \text{Gr}(d, N) \times \text{Gr}(r, M) \end{array}$$

in the category of topological spaces, where g is the product of the topological classifying maps for the bundles TX and E . Note that f and g are in general not analytic.

Let $V \rightarrow \text{Gr}(r, M)$ be the universal subbundle. If $\alpha \geq \mu$, let the bundle $V^{[\alpha]}$ on $\mathcal{Q}_{\text{Gr}} \times \text{Gr}(r, M)$ be defined by

$$V^{[\alpha]} = V \otimes \mathcal{O}_{\overline{U}^n}^{[\alpha]},$$

where V and $\mathcal{O}_{\overline{U}^n}^{[\alpha]}$ are pulled back from $\text{Gr}(r, M)$ and \mathcal{Q}_{Gr} , respectively. We then have $f^*(V^{[\alpha]}) = \mathcal{E}^{[\alpha]}$. The scheme \mathcal{Q}_{Gr} also carries a bundle $T_{\overline{U}^{[n]}}^{[\alpha]}$, defined in the same way as $T_{\overline{TX}^{[n]}}^{[\alpha]}$. It is easy to check that $f^*(T_{\overline{U}^{[n]}}^{[\alpha]}) = T_{\overline{TX}^{[n]}}^{[\alpha]}$.

For any $\alpha \geq \mu$, define the relative Nash bundle $T_{\tilde{\mathcal{T}}_{\alpha, \text{Gr}}/\text{Gr}}$ on \mathcal{Q}_{Gr} as the kernel of the natural map $T_{\tilde{\mathcal{T}}_{\alpha, \text{Gr}}} \rightarrow q^*(T_{\text{Gr}(d, N)})$. This map is surjective, as can be seen easily from the fact that $\mathcal{T}_{\alpha, \text{Gr}} \rightarrow \text{Gr}(d, N)$ is a locally trivial fibration. Hence there is a short exact sequence

$$0 \rightarrow T_{\tilde{\mathcal{T}}_{\alpha, \text{Gr}}/\text{Gr}} \rightarrow T_{\tilde{\mathcal{T}}_{\alpha, \text{Gr}}} \rightarrow q^*(T_{\text{Gr}(d, N)}) \rightarrow 0.$$

Similarly define $T_{\tilde{\mathcal{T}}_{\alpha}/X}$ by the short exact sequence

$$0 \rightarrow T_{\tilde{\mathcal{T}}_{\alpha}/X} \rightarrow T_{\tilde{\mathcal{T}}_{\alpha}} \rightarrow p^*(T_X) \rightarrow 0.$$

We then have $f^*(T_{\tilde{\mathcal{T}}_{\alpha, \text{Gr}}/\text{Gr}}) = T_{\tilde{\mathcal{T}}_{\alpha}/X}$. Define a bundle G_{α} on \mathcal{Q}_{Gr} by $G_{\alpha} = T_{\tilde{\mathcal{T}}_{\alpha, \text{Gr}}/\text{Gr}} \oplus q^*(U)$. In K-classes we then have $f^*(G_{\alpha}) = T_{\tilde{\mathcal{T}}_{\alpha}}$. Define the class

$\mathcal{C}_{\alpha, \text{Gr}} \in H^*(\mathcal{Q}_{\text{Gr}} \times \text{Gr}(r, M))$ by

$$\mathcal{C}_{\alpha, \text{Gr}} = F\left(\mathcal{E}_{\text{Gr}}^{[\alpha]}, T_{\overline{U}^{[n]}}^{[\alpha]}\right) \cdot c_{\bullet}(G_{\alpha}).$$

The above discussion shows that $f^*(\mathcal{C}_{\alpha, \text{Gr}}) = \mathcal{C}_{\alpha}$. It is easy to check that we have $p_!(\mathcal{C}_{\alpha}) = g^*((q \times \text{id})_!(\mathcal{C}_{\alpha, \text{Gr}}))$, which implies

$$\deg(\mathcal{C}_{\alpha} \cap [\mathcal{Q}]) = \deg(g^*((q \times \text{id})_!(\mathcal{C}_{\alpha, \text{Gr}})) \cap [X]).$$

Any rational cohomology class on $\text{Gr}(d, N) \times \text{Gr}(r, M)$ is a polynomial in Chern classes of the universal bundles. In particular, $(q \times \text{id})_!(\mathcal{C}_{\alpha, \text{Gr}})$ is equal to such a polynomial. The right hand side of the above equation is thus a linear combination of the Chern numbers of (X, E) .

It remains to be seen that this linear polynomial is independent of M and N . But this follows from the fact that for M and N sufficiently large, the class $\mathcal{C}_{\alpha, \text{Gr}}$ is preserved by the pullbacks induced by the natural morphisms $\text{Gr}(d, N) \rightarrow \text{Gr}(d, N+1)$ and $\text{Gr}(r, M) \rightarrow \text{Gr}(r, M+1)$.

Consequently, the degree of $\mathcal{C}_{\alpha} \cap [\mathcal{Q}]$ and thus that of \mathcal{C}_{α} is a universal linear combination of the Chern numbers of (X, E) . As \mathcal{D} is a universal linear combination of the \mathcal{C}_{α} , the claim follows. \square

5. PROOF OF LEMMA 4.19

5.1. Defining the Map from Q to \mathcal{Q} . Let $p_1, p_2 : X \times X \rightarrow X$ be the projection to the first and second factor, and let $\pi : TX \rightarrow X$ be the tangent bundle.

Lemma 5.1. *There is an open neighbourhood U_1 of the diagonal $\Delta \subset X \times X$, an open neighbourhood \mathcal{U}_1 of the 0-section $X \subset TX$ and a homeomorphism $f_1 : U_1 \rightarrow \mathcal{U}_1$, such that*

$$\pi \circ f_1 = p_1$$

and such that $f_1|_{\Delta}$ is the natural identity map between Δ and the 0-section of TX . Furthermore, the restriction of f_1 to each fibre $p_1^{-1}(x)$ is holomorphic. There is an isomorphism of topological vector bundles $p_1^(E)|_{U_1} \rightarrow p_2^*(E)|_{U_1}$, which is an isomorphism of holomorphic bundles on the restriction to each fibre $p_1^{-1}(x)$.*

Proof. See [Li, Lemma 2.4] for the first statement. Essentially, (the inverse of) holomorphic exponential maps can be constructed on small open sets, and these can be globalized using a partition of unity. Globalizing breaks analyticity of the map, but preserves it on fibres of p_1 as required.

For the statement about E , we argue similarly. Cover X with open balls B_i , and choose holomorphic trivialization $E|_{B_i} \cong \mathcal{O}_{B_i}^n$. Using these, define local isomorphisms $g_i : p_1^*(E|_{B_i}) \rightarrow p_2^*(E|_{B_i})$. Let $\{t_i\}$ be a partition of unity subordinate to the covering $\{B_i\}$, and define $g : p_1^*(E) \rightarrow p_2^*(E)$ at $x \in U_1$ by $g(x) = \sum t_i(p_1(x)) \cdot g_i$.

It is easy to check that g is holomorphic on fibres of p_1 . Restricted to Δ , the map g is the identity, and so shrinking U_1 if necessary, g is an isomorphism. \square

Let $X_0^{[n]} \subset X^{[n]}$ be the set of points $(Z, (x_i))$ such that Z is supported at a single point, and let $\overline{TX}_0^{[n]} \subset \overline{TX}^{[n]}$ denote the set of points $(Z, (x_i))$ such that Z is supported at the 0-section of \overline{TX} . Let $q : X^{[n]} \rightarrow X$ be defined by $q(Z, (x_i)) = x_1$, and let $r : \overline{TX}^{[n]} \rightarrow \overline{TX}$ be defined similarly. Let W be the set of pairs $(Z, (x_i)) \in \overline{TX}^{[n]}$ such that x_1 lies in the 0-section of \overline{TX} .

Lemma 5.2. *There is an open neighbourhood U_2 of $X_0^{[n]}$ in $X^{[n]}$, an open neighbourhood \mathcal{U}_2 of $\overline{TX}_0^{[n]}$ in W , and a homeomorphism $f_2 : U_2 \rightarrow \mathcal{U}_2$, such that*

$$q = \pi \circ r \circ f_2.$$

Furthermore, for every point $x \in X$, the restriction of f_2 to a map $q^{-1}(x) \rightarrow r^{-1}(x)$ is holomorphic. There are isomorphisms of topological vector bundles $f_2^*(T_{\overline{TX}^{[n]}}^{[n]}) \rightarrow T_{X^{[n]}}^{[n]}$ and $f_2^*(\mathcal{E}^{[n]}) \rightarrow E^{[n]}$.

Proof. Let U_1, \mathcal{U}_1 and f_1 be as provided by Lemma 5.1. In every fibre $q^{-1}(x) \subset X^{[n]}$ there is an open subset U_x of $q^{-1}(x)$ defined by the condition that the support of the subscheme is contained in $U_1 \cap \{x\} \times X$. Define the map f_2 on $U_x^{[n]}$ by $f_2((Z, (x_i))) = ((f_1)_*(Z \times \{x\}), (f_1((x_i), x)))$. This is a local homeomorphism which is analytic as a map $q^{-1}(x) \rightarrow r^{-1}(x)$.

It is easy to check from the definition of f_2 that the natural morphism $T_{X^{[n]}}^{[n]} \rightarrow f_2^*(T_{\overline{TX}^{[n]}}^{[n]})$ is an isomorphism, and the isomorphism $f_1^*(\pi^*(E)) = p_1^*(E) \rightarrow p_2^*(E)$ of Lemma 5.1 induces an isomorphism of topological vector bundles $f_2^*(\mathcal{E}^{[n]}) \rightarrow E^{[n]}$. \square

Recall that $Q_0 \subset Q$ and $\mathcal{Q}_0 \subset \mathcal{Q}$ are the subsets of points having image in $\Delta \subset X^n$ and $X \subset \overline{TX}^n$ under the natural morphisms $Q \rightarrow X^n$ and $\mathcal{Q} \rightarrow \overline{TX}^n$, respectively.

Let the relative Nash bundles $T_{\widetilde{T}_\alpha/X}$ and $T_{\widetilde{\mathcal{T}}_\alpha/X}$ be the kernels of the surjective homomorphisms $T_{\widetilde{T}_\alpha} \rightarrow q^{-1}(T_X)$ and $T_{\widetilde{\mathcal{T}}_\alpha} \rightarrow r^{-1}(T_X)$, respectively.

Lemma 5.3. *There is an open neighbourhood U of Q_0 in Q , an open neighbourhood \mathcal{U} of \mathcal{Q}_0 in \mathcal{Q} , and a homeomorphism $f : U \rightarrow \mathcal{U}$, as well as homomorphisms*

$$f^*(\mathcal{E}^{[\alpha]}) \rightarrow E^{[\alpha]},$$

$$f^*(T_{\overline{TX}^{[n]}}^{[\alpha]}) \rightarrow T_{X^{[n]}}^{[\alpha]},$$

and

$$f^*(T_{\widetilde{T}_\alpha/X}) \rightarrow T_{\widetilde{\mathcal{T}}_\alpha/X},$$

which are isomorphisms of topological vector bundles on U .

Proof. The proof of the previous lemma can easily be modified to show that there are local isomorphisms $T_\alpha \rightarrow \mathcal{T}_\alpha$. The Nash blow up is determined analytically locally, and $T_\alpha \xrightarrow{q} X$ and $\mathcal{T}_\alpha \xrightarrow{r} X$ are both locally trivial fibrations in a neighbourhood of Q_0 and \mathcal{Q}_0 . Hence the Nash blow ups are locally fibrations as well. Using this, it is easy to see that one gets local homeomorphisms $\widetilde{T}_\alpha \rightarrow \widetilde{\mathcal{T}}_\alpha$ agreeing with the maps $T_\alpha \rightarrow \mathcal{T}_\alpha$ on the nonsingular loci of T_α and \mathcal{T}_α . This in turn induces a local isomorphism $Q \rightarrow \mathcal{Q}$. The first bundle isomorphism follows from the one produced in Lemma 5.2, and taking into account the fact that $T_\alpha \rightarrow \mathcal{T}_\alpha$ is holomorphic on the fibres of q and r , we get the second and third. \square

5.2. Construction of \overline{D} , $\overline{\mathcal{D}}$. In order to prove the claim of Lemma 4.19, we will construct the relative cohomology classes $\overline{D} \in H^*(Q, Q \setminus U)$ and $\overline{\mathcal{D}} \in H^*(\mathcal{Q}, \mathcal{Q} \setminus V)$ explicitly as singular cochains. We first define certain open subsets $U_\alpha, V_{\alpha,\beta}$ of X^n , which we will need in order to compare the bundles $E^{[\alpha]} (T_{X^{[n]}}^{[\alpha]}, T_{\widetilde{T}_\alpha})$ for various α .

Let d_X be a metric on X inducing the Euclidean topology. Define a metric d_{X^n} on X^n by

$$d_{X^n}((x_i)_{i=1}^n, (y_i)_{i=1}^n) = \max_{1 \leq i \leq n} d_X(x_i, y_i).$$

Let $d_{\overline{TX}}$ be a metric on \overline{TX} inducing the Euclidean topology, and let $d_{\overline{TX}^n}$ be the metric on \overline{TX}^n defined similarly to d_{X^n} .

Let $x \in X^n$ and let α be a partition. Recall that $\Delta_\alpha \subseteq X^n$ is the diagonal set

$$\{(x_1, \dots, x_n) \mid x_i = x_j \text{ if } i \sim_\alpha j\},$$

and let $\Delta'_\alpha \subset \overline{TX}^n$ be defined in the same way. In the following, we will use the inequalities

$$(1) \quad \frac{1}{2} \sup_{i,j| i \sim_{\alpha} j} d_X(x_i, x_j) \leq d_{X^n}(x, \Delta_\alpha) \leq \sup_{i,j| i \sim_{\alpha} j} d_X(x_i, x_j)$$

and their variants for $d_{\overline{TX}}$ and $d_{\overline{TX}^n}$, all of which follow easily from the definitions and the triangle inequality.

Definition 5.4. Let $P(n)$ be the set of partitions of n , and let $\epsilon : P(n) \rightarrow \mathbb{R}^{>0}$ be a function such that the quantities

$$\max_{\alpha \in P(n)} \epsilon(\alpha)$$

and

$$\max_{\alpha < \beta \in P(n)} \frac{\epsilon(\alpha)}{\epsilon(\beta)}$$

are both sufficiently small.

Definition 5.5. Let $U_\alpha \subset X^n$ be the open $\epsilon(\alpha)$ -neighbourhood of $\Delta_\alpha \subset X^n$, and let \mathcal{U}_α be the open $\epsilon(\alpha)$ -neighbourhood of $\Delta'_\alpha \subset \overline{TX}^n$.

Definition 5.6. Let α, β be partitions such that $\alpha \leq \beta$. Define the set $V_{\alpha, \beta} \subset X^n$ as

$$V_{\alpha, \beta} = (X^n \setminus U_\beta) \setminus \left(\bigcup_{\substack{\gamma \leq \beta \\ \gamma \not\leq \alpha}} \overline{U}_\gamma \right),$$

and define the set $\mathcal{V}_{\alpha, \beta} \subset \overline{TX}^n$ as

$$\mathcal{V}_{\alpha, \beta} = (\overline{TX}^n \setminus \mathcal{U}_\beta) \setminus \left(\bigcup_{\substack{\gamma \leq \beta \\ \gamma \not\leq \alpha}} \overline{\mathcal{U}}_\gamma \right).$$

Let $i, j \in \{1, \dots, n\}$. Define an equivalence relation $\sim_{(i,j)}$ on the set of partitions by saying $\alpha \sim_{(i,j)} \beta$ if \sim_α and \sim_β agree when evaluated on the pair (i, j) . For any pair of partitions α, β , define the set $\Delta_{\alpha\beta} \subseteq X^n$ to be the set of points over which $X^{\llbracket \alpha \rrbracket}$ and $X^{\llbracket \beta \rrbracket}$ are not canonically equal. Explicitly, we have

$$\Delta_{\alpha\beta} = \bigcup_{i,j| \alpha \not\sim_{(i,j)} \beta} \Delta_{ij},$$

where Δ_{ij} denotes the set of points $x \in X^n$ for which $x_i = x_j$. Define $\Delta'_{\alpha\beta} \subset \overline{TX}^n$ similarly.

The following lemma summarizes the important properties of $V_{\alpha, \beta}$ and $\mathcal{V}_{\alpha, \beta}$.

Lemma 5.7.

- (i) Let β be a partition. The sets $\{V_{\alpha, \beta}\}_{\alpha < \beta}$ form an open covering of $X^n \setminus U_\beta$.
The sets $\{\mathcal{V}_{\alpha, \beta}\}_{\alpha < \beta}$ form an open covering of $(\overline{TX}^n) \setminus \mathcal{U}_\beta$.
- (ii) Let α, β, γ be partitions such that $\alpha, \gamma \leq \beta$ and $\gamma \not\leq \alpha$. Then

$$U_\gamma \cap V_{\alpha, \beta} = \emptyset \quad \text{and} \quad \mathcal{U}_\gamma \cap \mathcal{V}_{\alpha, \beta} = \emptyset.$$

- (iii) Let $\tau = \min_\gamma \epsilon(\gamma)$, and let $\alpha < \beta$. If $x \in V_{\alpha, \beta}$, we have $d(x, \Delta_{\alpha\beta}) \geq \tau$. If $x \in \mathcal{V}_{\alpha, \beta}$, we have $d(x, \Delta'_{\alpha\beta}) \geq \tau$.

Proof. We prove the statements for $V_{\alpha,\beta}$; the case of $\mathcal{V}_{\alpha,\beta}$ is exactly the same.

(i): Assume $x = (x_i) \in X^n \setminus U_\beta$, and let α be maximal among partitions $< \beta$ such that $x \in \overline{U_\alpha}$. Such a partition exists because for the smallest partition ω , we have $U_\omega = X^n$. We claim that $x \in V_{\alpha,\beta}$.

Assume $x \notin V_{\alpha,\beta}$, there is then a partition γ such that $\gamma \not\leq \alpha$, $\gamma < \beta$ and $x \in \overline{U_\gamma}$. By the maximality property of α , we cannot have $\alpha < \gamma$. It follows that $\alpha, \gamma < (\alpha \vee \gamma) \leq \beta$, where $\alpha \vee \gamma$ is the smallest partition majorizing α and γ .

Let i, j be two indices such that $i \sim_{\alpha \vee \gamma} j$, and such that $d(x_i, x_j)$ is maximal among pairs with this property. There is a sequence of integers i_1, i_2, \dots, i_r such that $i_1 = i$, $i_r = j$ and such that for every k with $1 \leq k < r$, either $i_k \sim_\alpha i_{k+1}$ or $i_k \sim_\gamma i_{k+1}$ is true. By (1), we now have

$$\begin{aligned} d(x, \Delta_{\alpha \vee \gamma}) &\leq d(x_i, x_j) \leq d(x_{i_1}, x_{i_2}) + \dots + d(x_{i_{r-1}}, x_{i_r}) \\ &\leq (r-1)(2\epsilon(\alpha) + 2\epsilon(\gamma)) \leq 2n(\epsilon(\alpha) + \epsilon(\gamma)) < \epsilon(\alpha \vee \gamma), \end{aligned}$$

where the last step uses the second condition in the definition of ϵ . Hence we have $x \in U_{\alpha \vee \gamma}$, which contradicts either the maximality of α or the fact that $x \notin U_\beta$.

(ii): Obvious from the definition.

(iii): Let $x = (x_i) \in V_{\alpha,\beta}$. For every $i, j \in \{1, \dots, n\}$, let $\gamma_{i,j}$ be the partition defined by the equivalence relation such that $i \sim_{\gamma_{i,j}} j$ and no other nontrivial relations hold. If $\alpha < \beta$, we have

$$\Delta_{\alpha\beta} = \bigcup_{\substack{i \sim_\beta j \\ i \not\sim_\alpha j}} \Delta_{\gamma_{i,j}}.$$

For every pair i, j occurring in the union, we have $\gamma_{i,j} \not\leq \alpha$, hence by part (ii) of the lemma we have $x \notin U_{\gamma_{i,j}}$. This gives

$$d(x, \Delta_{\alpha\beta}) = \min_{\substack{i \sim_\beta j \\ i \not\sim_\alpha j}} d(x, \Delta_{\gamma_{i,j}}) > \min_{\substack{i \sim_\beta j \\ i \not\sim_\alpha j}} \epsilon(\gamma_{i,j}) \geq \tau.$$

□

Recall that $T_{\tilde{T}_\alpha/X}$ and $T_{\tilde{T}_\alpha/X}$ are the relative Nash bundles defined above Lemma 5.3. Denote by \mathcal{O}^N the trivial bundle of rank N .

Lemma 5.8.

- (i) For each $\alpha \geq \mu$, let F_α denote either $E^{[\alpha]}$, $T_{\tilde{T}_\alpha/X}$ or $T_{X^{[n]}}^{[\alpha]}$, considered as a topological bundle on Q . There is an integer M and for every α an injection $i_\alpha : F_\alpha \rightarrow \mathcal{O}^M$ such that if $\alpha \leq \beta$, then over $V_{\alpha,\beta}$ the bundles $i_\alpha(E^{[\alpha]})$ and $i_\beta(E^{[\alpha]})$ are equal as subbundles of \mathcal{O}^M .
- (ii) For each $\alpha \geq \mu$, let \mathcal{F}_α denote either $\mathcal{E}^{[\alpha]}$, $T_{\tilde{T}_\alpha/X}$ or $T_{\tilde{T}_X^{[n]}}^{[\alpha]}$, considered as a topological bundle on \mathcal{Q} . There is an integer N and for every α an injection $j_\alpha : \mathcal{F}_\alpha \rightarrow \mathcal{O}^N$ such that if $\alpha \leq \beta$, then over $\mathcal{V}_{\alpha,\beta}$ the bundles $j_\alpha(\mathcal{F}^{[\alpha]})$ and $j_\beta(\mathcal{F}^{[\alpha]})$ are equal as subbundles of \mathcal{O}^N .
- (iii) Let $f : U \rightarrow \mathcal{U}$ be the local homeomorphism constructed in Lemma 5.3. We may choose $M = N$ and the injections i_α, j_α in such a way that the diagram

$$\begin{array}{ccc} f^*(\mathcal{F}_\alpha) & \xleftarrow{\cong} & F_\alpha \\ f^*(j_\alpha) \downarrow & & i_\alpha \downarrow \\ f^*(\mathcal{O}^N) & \xlongequal{\quad} & \mathcal{O}^N \end{array}$$

of bundles on U commutes, where the upper isomorphism is the one constructed in Lemma 5.3.

Proof. In this proof, all partitions are assumed to be $\geq \mu$.

(i): For each α choose first an injective homomorphism $i'_\alpha : F_\alpha \rightarrow \mathcal{O}^{M_\alpha}$. Let $k_{\alpha\beta} : F_\alpha \rightarrow F_\beta$ be the natural isomorphisms, defined over $X^n \setminus \Delta_{\alpha\beta}$. Recall that $\tau = \min_\alpha \epsilon(\alpha)$. Let $t : \mathbb{R}^{\geq 0} \rightarrow \mathbb{R}^{\geq 0}$ be a continuous function such that when $x \geq \tau$ we have $t(x) = 1$ and such that when $x \leq \tau/2$ we have $t(x) = 0$. For $x \in Q$, put $t_{ij}(x) = t(d(p(x), \Delta_{ij}))$, where $p : Q \rightarrow X^n$ is the natural morphism. Let

$$t_{\alpha\beta} = \prod_{i \leq j | \alpha \not\prec_{(i,j)} \beta} t_{ij}.$$

As $t_{\alpha\beta}$ is supported away from $\Delta_{\alpha\beta}$, the homomorphism $t_{\alpha\beta} \cdot k_{\alpha\beta}$ is defined on the whole of Q .

For any two partitions α and γ , let $i_{\alpha\gamma} = i'_\gamma \circ (t_{\alpha\gamma} \cdot k_{\alpha\gamma})$. We take $M = \sum M_\gamma$, and set

$$i_\alpha = (i_{\alpha\gamma})_\gamma : E_\alpha \rightarrow \oplus \mathbb{C}^{M_\gamma} = \mathbb{C}^M.$$

It remains to show that i_α has the properties stated. As $i_{\alpha\alpha} = i'_\alpha$ is injective, it is clear that i_α is an injection.

Let $\alpha < \beta$, and let γ be arbitrary. First we show that if $x \in Q$ lies over $V_{\alpha,\beta}$, then

$$(2) \quad (t_{\alpha\gamma} \cdot k_{\alpha\gamma})(x) = (t_{\beta\gamma} \cdot k_{\beta\gamma} \circ k_{\alpha\beta})(x).$$

To this end, observe that

$$\frac{t_{\alpha\gamma}}{t_{\beta\gamma}} = \prod_{i \leq j | \alpha \not\prec_{(i,j)} \beta} t_{ij}^{\delta(i,j)},$$

where each $\delta(i,j) \in \{-1, 0, 1\}$. Now, since $p(x) \in V_{\alpha,\beta}$, we have $t_{ij}(x) = 1$ for every factor above, using Lemma 5.7 (iii). Hence, the equation $t_{\alpha\gamma}(x) = t_{\beta\gamma}(x)$ holds. If $t_{\alpha\gamma} = 0$, this shows (2). If not, then all the morphisms $k_{\alpha\beta}, k_{\beta\gamma}, k_{\alpha\gamma}$ are defined at x , and by the naturality of these the cocycle condition $k_{\alpha\gamma} = k_{\beta\gamma} \circ k_{\alpha\beta}$ holds.

The above paragraph shows that $i_{\alpha\gamma} = i_{\beta\gamma} \circ k_{\alpha\beta}$ over $V_{\alpha,\beta}$ and hence $i_\alpha = i_\beta \circ k_{\alpha\beta}$. Consequently the two subbundles $i_\alpha(E_\alpha)$ and $i_\beta(E_\beta)$ of \mathbb{C}^M are equal as claimed.

(ii): Similar to (i).

(iii): Let $k'_{\alpha\beta} : \mathcal{F}_\alpha \rightarrow \mathcal{F}_\beta$ be the homomorphisms defined like the $k_{\alpha\beta}$ in the proof of (i). Let $g_\alpha : F_\alpha \rightarrow f^*(\mathcal{F}_\alpha)$ be the isomorphism of Lemma 5.3. It is easy to see that we then have $g_\beta \circ k_{\alpha\beta} = f^*(k'_{\alpha\beta}) \circ g_\alpha$.

Let $U' \subset U$ and $\mathcal{U}' \subset \mathcal{U}$ be smaller open neighbourhoods of Q_0 and \mathcal{Q}_0 , such that $f(U') = \mathcal{U}'$. Let s be a nonnegative real function which is 1 on U' and 0 on the complement of U . Replace i_α with

$$i'_\alpha := ((1-s)i_\alpha, s \cdot f^*(j_\alpha) \circ g_\alpha) : F_\alpha \rightarrow \mathcal{O}^{M+N},$$

where $j_\alpha : \mathcal{F}_\alpha \rightarrow \mathcal{O}^N$ is the homomorphism of part (ii). Replacing j_α with $j'_\alpha = (0, j_\alpha) \rightarrow \mathcal{O}^M \oplus \mathcal{O}^N$, then obviously $i'_\alpha = f^*(j'_\alpha) \circ g_\alpha$ over U' . After replacing U with U' , it now remains to be shown that the statement in part (i) of the lemma still holds for i'_α .

After enlarging the metric on \overline{TX} and shrinking U and \mathcal{U} , we may assume that if $x \in U$ lies over $V_{\alpha,\beta}$, then $f(x)$ lies over $\mathcal{V}_{\alpha,\beta}$. Over $V_{\alpha,\beta}$ we thus have

$$\begin{aligned} i'_\alpha &= ((1-s)i_\alpha, s \cdot f^*(j_\alpha) \circ g_\alpha) = ((1-s)i_\beta \circ k_{\alpha\beta}, s \cdot f^*(j_\beta \circ k'_{\alpha\beta}) \circ g_\alpha) \\ &= ((1-s)i_\beta, s \cdot f^*(j_\beta) \circ g_\beta) \circ k_{\alpha\beta} = i'_\beta \circ k_{\alpha\beta}. \end{aligned}$$

□

We are now in a position to finish the proof of Lemma 4.19.

Proof of Lemma 4.19: Let α be a partition $\geq \mu$. The inclusions produced in Lemma 5.7 (ii) define a continuous map $f_\alpha : \mathcal{Q} \rightarrow \text{Gr}_1 \times \text{Gr}_2 \times \text{Gr}_3$, where the Gr_i are Grassmannians with universal bundles W_i , such that

$$f_\alpha^*(W_1) = \mathcal{E}^{[\alpha]} \quad f_\alpha^*(W_2) = T_{\overline{TX}^{[n]}}^{[\alpha]}, \quad f_\alpha^*(W_3) = T_{\widetilde{\mathcal{T}_\alpha/X}} \oplus r^*(T_X).$$

Note that at the level of Chern classes, we may interchange $f_\alpha^*(W_3)$ with $T_{\widetilde{\mathcal{T}_\alpha}}$, as these are topologically isomorphic.

Choose a singular cochain A on $\text{Gr}_1 \times \text{Gr}_2 \times \text{Gr}_3$ representing the class

$$F(W_1, W_2) \cdot c_\bullet(W_3).$$

Define singular cochains $\overline{\mathcal{C}}_\alpha$ and $\overline{\mathcal{D}}_\alpha$ by $\overline{\mathcal{C}}_\alpha = f_\alpha^*(A)$ and

$$\overline{\mathcal{D}}_\alpha = \overline{\mathcal{C}}_\alpha - \sum_{\gamma \in [\mu, \alpha)} \overline{\mathcal{D}}_\gamma.$$

Clearly, the class of $\overline{\mathcal{C}}_\alpha$ and $\overline{\mathcal{D}}_\alpha$ is \mathcal{C}_α and \mathcal{D}_α , respectively. Let $\overline{\mathcal{C}} = \overline{\mathcal{C}}_\Lambda$ and $\overline{\mathcal{D}} = \overline{\mathcal{D}}_\Lambda$.

We now claim that $\overline{\mathcal{D}}|_{\overline{TX}^n \setminus \mathcal{U}_\Lambda} = 0$. To prove this, we show that $\overline{\mathcal{D}}_\beta|_{\overline{TX}^n \setminus \mathcal{U}_\beta} = 0$ for any partition β by ascending induction on the ordering of partitions. The base case is clear, as $\mathcal{Q}|_{\overline{TX}^n \setminus \mathcal{U}_\mu} = \emptyset$.

Assume now that $\overline{\mathcal{D}}_\alpha|_{\overline{TX}^n \setminus \mathcal{U}_\alpha} = 0$ for every $\alpha < \beta$. We must show that for every singular m -simplex $a : \Delta^m \rightarrow \mathcal{Q}|_{\overline{TX}^n \setminus \mathcal{U}_\beta}$ we have $\overline{\mathcal{D}}_\beta(a) = 0$.

Since $\overline{\mathcal{D}}_\beta$ is a cocycle, we may replace a by any subdivision of a and prove the vanishing for each simplex in the subdivision. By Lemma 5.7 (i), $\{\mathcal{V}_{\alpha, \beta}\}_{\alpha < \beta}$ is an open covering of $\overline{TX}^n \setminus \mathcal{U}_\beta$, so we may assume there is an $\alpha < \beta$ such that a maps into $\mathcal{Q}|_{\mathcal{V}_{\alpha, \beta}}$.

If $\gamma < \beta$ is such that $\gamma \not\leq \alpha$, then by Lemma 5.7 (ii) we have $\mathcal{U}_\gamma \cap \mathcal{V}_{\alpha, \beta} = \emptyset$, and so by the induction hypothesis $\overline{\mathcal{D}}_\gamma(a) = 0$. This implies

$$\overline{\mathcal{D}}_\beta(a) = \overline{\mathcal{C}}_\beta(a) - \sum_{\gamma \leq \alpha} \overline{\mathcal{D}}_\gamma(a) = \overline{\mathcal{C}}_\beta(a) - \overline{\mathcal{C}}_\alpha(a),$$

where the last equality follows directly from the definition of $\overline{\mathcal{D}}_\alpha$. By Lemma 5.8 (ii) we have $f_\alpha = f_\beta$ over $\mathcal{V}_{\alpha, \beta}$, hence $\overline{\mathcal{D}}_\beta(a) = 0$ and the claim follows.

Taking ϵ small enough we may assume that $\mathcal{Q} \setminus \mathcal{U}'$ lies over $\overline{TX}^n \setminus \mathcal{U}_\Lambda$, and then the above shows $\overline{\mathcal{D}}$ is a relative cocycle for the pair $(\mathcal{Q}, \mathcal{Q} \setminus \mathcal{U}')$. The exact same construction performed on \mathcal{Q} produces a cochain $\overline{\mathcal{D}}_\alpha$ of class \mathcal{D}_α which vanishes on $\mathcal{Q} \setminus \mathcal{U}'$. By Lemma 5.8 (iii), the homomorphism

$$H^*(\mathcal{Q}, \mathcal{Q} \setminus \mathcal{U}') \rightarrow H^*(\mathcal{U}, \mathcal{U} \setminus \mathcal{U}') \xrightarrow{f^*} H^*(U, U \setminus U') \rightarrow H^*(Q, Q \setminus U')$$

sends $\overline{\mathcal{D}}$ to $\overline{\mathcal{D}}$, which proves Lemma 4.19. \square

This concludes the proof of Theorems 1.1 and 1.2.

6. A GENERATING FUNCTION

As was noted in [EGL] and elsewhere, the existence of universal polynomials often implies a stronger statement about the form of the generating function of all Chern integrals.

Definition 6.1. Let X be as in the main theorem, let Q_1, \dots, Q_k be closed, irreducible punctual geometric subsets with $Q_i \subseteq \text{Hilb}_0^{n_i}(\mathbb{C}^d)$, and let m_1, \dots, m_k be nonnegative integers. Let (Q'_i) be a sequence of length $\sum m_i$ with each Q_i appearing m_i times. Let $n = m_1 n_1 + \dots + m_k n_k$, and define the set

$$\overline{P}(Q_1^{m_1}, \dots, Q_k^{m_k}) := \overline{P}((Q'_i)_{i=1}^n) \subseteq X^{[n]},$$

in the notation of Definition 2.4.

In other words, the set $\overline{P(Q_1^{m_1}, \dots, Q_k^{m_k})}$ is the closure of the set of Z such that Z is the disjoint union of m_1 subschemes from Q_1 , m_2 subschemes from Q_2 , and so on.

Let $CM(k)$ denote the monoid of all k -variable Chern monomials (including the unit monomial) under multiplication. Theorem 1.1 now yields the following result.

Proposition 6.2. *Let X be a smooth, connected projective variety of dimension d , let E_1, \dots, E_k be algebraic vector bundles and let Q_1, \dots, Q_l be closed, irreducible punctual geometric subsets such that the Q_i are pairwise distinct. Let R be the ring $\mathbb{Q}[[x, y_1, \dots, y_l]]$, where the x -variable is graded by $CM(k)$. Then, for every $N \in CM(k+1, d)$ there exists a $B_N \in R$, depending only on the Q_i and the ranks of the E_i , such that*

$$\sum_{\substack{M \in CM(k) \\ m_i \geq 0}} \deg M \left((E_i^{[n]})_{i=1}^k \right) \cap [P((Q_j^{m_j})_{j=1}^l)] x^M y_1^{m_1} \cdots y_l^{m_l} = \prod_{N \in CM(k+1, d)} B_N^{N(T_X, E_1, \dots, E_k)}.$$

If $\dim X$ is 1 or 2, the same statement holds with $M((E_i^{[n]}))$ replaced by $M(T_{X^{[n]}}, (E_i^{[n]}))$ and $CM(k)$ replaced with $CM(k+1)$.

Proof. We treat the case without the $T_{X^{[n]}}$, the case including $T_{X^{[n]}}$ is essentially the same.

Let f be the generating function on the left hand side of the equation in the proposition. We show that the logarithm of f is linear in the Chern numbers of $(X, (E_i))$, hence it is of the form

$$\log f = \sum_{N \in CM(k+1, d)} N(T_X, (E_i)) b_N$$

for elements $b_N \in R$. Taking $B_N = \exp b_N$ then yields the proposition.

The fact that $\log f$ is linear is a consequence of the linearity of the Chern polynomial evaluating $\deg D \cap [T]$, where D and T are as in the proof of the main theorem. Fix integers m_i , let $P = [P((Q_j^{m_j}))]$, and let T be a set covering P as in the proof of the main theorem.

Using the irreducibility and distinctness of the Q_i , it is easy to show that there are points $Z_i \in Q_i$ for every i , such that Z_i is not isomorphic to a point in Q_j if $i \neq j$. For a point in P isomorphic to m_i disjoint copies of each Z_i , there are $\prod m_i!$ points in T lying over it, which implies that $\deg T/P = \prod m_i!$.

Now, let

$$G = \sum_M \deg M((E_i^{[n]})) \cap [T] x^M.$$

Let μ be the partition of n as in the main proof, and for every partition $\alpha \geq \mu$, let

$$G_\alpha = \sum_M \deg M((E_i^{[\alpha]})) \cap [T_\alpha] x^M,$$

where T_α is as in the proof of the main theorem.

For any set $A \subseteq \{1, \dots, n\}$ such that A is a union of sets from μ , let $A_i \subset A$ be the set of labels corresponding to subschemes isomorphic to $Z \in Q_i$ in the labelling induced by T . Suppose n_i is such that $Q_i \in \text{Hilb}^{n_i}(\mathbb{C}^d)$, and let $m_{i,A} = |A_i|/n_i$. Let $T_A \subset X^{[A]}$ be the scheme covering $P_A = \overline{P((Q_i^{m_{i,A}}))}$. In the notation of the main proof, we then have $T_\alpha = \prod_{A \in \alpha} T_A$. Let

$$G_A = \sum_M \deg M((E^{[|A|]})) \cap [T_A] x^M,$$

the argument of the proof of Lemma 4.13 then shows

$$G_\alpha = \prod_{A \in \alpha} G_A.$$

Let now $C_{\alpha,M} = M((E_i^{\llbracket \alpha \rrbracket}))$, and let D_M be defined in terms of these as in Definitions 4.11 and 4.12. By the remark following these definitions, we have

$$D_M = \sum_{\alpha \geq \mu} (-1)^{|\alpha|-1} (|\alpha| - 1)! C_{\alpha,M}.$$

We thus let

$$H = \sum_{\alpha \geq \mu} (-1)^{|\alpha|-1} (|\alpha| - 1)! G_\alpha.$$

As each D_M is linear in the Chern numbers of $(X, (E_i))$, the same will be true for the coefficients of x^M in H . We claim that for every M , the coefficient of x^M in H equals that of the $x^M y_1^{m_1} \dots y_k^{m_k}$ -term of $\prod m_i! \log f$. Using $G_\alpha = \prod_{A \in \alpha} G_A$, together with the fact that G_A is $\prod_i m_{i,A}!$ times the sum of the $x^M y_1^{m_{1,A}} \dots y_k^{m_{k,A}}$ terms of f for different M , it is now a combinatorial exercise to match up the terms of H and $\prod m_i! \log f$. \square

7. ENUMERATIVE APPLICATIONS

We present some applications of the main theorem to the problem of counting geometric objects with prescribed singularities. We treat three different problems. In Section 7.1 we study curves on a surface having prescribed singularity type, where by singularity type we mean either analytic or topological (equisingular) type. If L is a sufficiently ample line bundle on a surface S , we show that the number of such curves in a general linear system $\mathbb{P}^d \subset |L|$ of appropriate dimension is given by a universal polynomial in the Chern numbers of (S, L) . A somewhat more general result to this effect has recently been obtained independently by Li and Tzeng in [LT], using a similar method.

In Section 7.2, we consider divisors having fixed isolated analytic singularity types on a smooth variety X of arbitrary dimension. We show that the number of such in a general linear system $\mathbb{P}^d \subset |L|$ is universal. The proofs carry over verbatim from the analytic curve singularity case and are omitted.

Finally, in section 7.3, we consider again the case of curves on a surface. We study the locus $|L|_m \subset |L|$ of curves having prescribed “BPS spectrum” m and show that if L is sufficiently ample, the Euler characteristic of $|L|_m \cap \mathbb{P}^k$ is universal.

All of our results use the assumption that L is sufficiently ample. This is required to ensure that the objects we consider occur in the expected codimension in $|L|$, as well as in other places in the argument. A natural way of measuring the ampleness of a line bundle in this setting is N -very ampleness, defined as follows.

Definition 7.1. Let X be a nonsingular, projective variety, and let L be a line bundle on X . We say that L is N -very ample if for every length $(N+1)$ -subscheme $Z \subset X$, the map $H^0(X, L) \rightarrow H^0(Z, L|_Z)$ is surjective.

Equivalently, L is N -very ample if the homomorphism of bundles $H^0(X, L) \otimes \mathcal{O}_{X^{[N+1]}} \rightarrow L^{[N+1]}$ is surjective.

7.1. Curves with Specified Singularities. We begin by fixing some terms. By a curve singularity we mean a pair (C, p) where C is a reduced, locally planar algebraic curve and p is a singular point of C .

Let (C, p) be a curve singularity. By the formal type of the singularity (C, p) we mean the isomorphism type of the complete local \mathbb{C} -algebra $\hat{\mathcal{O}}_{C,p}$. By the topological type (synonymously, equisingularity type) of (C, p) with C embedded in a

smooth surface S , we mean the homeomorphism type of the pair $(B_\epsilon(p), C \cap B_\epsilon(p))$, where $B_\epsilon(p)$ is a sufficiently small open ball in S centred at p .

Proposition 7.2. *Let S be a smooth, projective, connected surface, let L be a line bundle on S , and let T_1, \dots, T_k be analytic singularity types. Suppose L is N -very ample, where N is an integer depending on the types T_i , and let d be the sum of the codimensions of the T_i . There is a rational polynomial G of degree k in 4 variables, depending only on the T_i , such that in a general $\mathbb{P}^d \subseteq |L|$ the number of curves having precisely k singularities of types T_i is*

$$G(c_1^2(L), c_1(L)c_1(S), c_2(S), c_1^2(S)).$$

The same statement holds when the T_i are topological types.

A similar proposition has recently been obtained by Li and Tzeng in [LT]. They additionally treat the case where the types are such that some are analytic and some are topological.

Remark. We will not be concerned with the range of validity for the universal polynomial, and refrain from making the N in the statement of the proposition explicit. Instead, we will simply take N large enough whenever the N -very ampleness of L is required, it will be clear that N depends only on the types T_i . A value for N may be recovered from the proof, but already in the case of nodal singularities, the N provided by this method is known to be larger than required by a factor of 5; compare our model [Gö] with [KST].

The main idea of the proof, taken from [Gö], is to set up a correspondence between curves having given singularities and curves containing 0-dimensional subschemes of given isomorphism type.

Choosing analytic or topological singularity types T_i , we show that we may identify curves with such singularities by length m subschemes. We define a geometric set $W = W((T_i)) \subset S^{[m]}$ such that (generically) a curve containing a $Z \in W$ will have the specified singularities. Then, using a proposition from [Gö] we get that in a general $\mathbb{P}^d \subseteq |L|$, the number of curves containing a subscheme $Z \in W$ equals $\deg(c_{\dim W}(L^{[m]}) \cap [\overline{W}])$, which is universal by Theorem 1.1. We then show that there is a bijection between pairs (Z, C) and curves in \mathbb{P}^d with the T_i as singularity types, completing the proof.

Corollary 7.3. *Let T_1, \dots, T_k be distinct analytic singularity types, and let n_1, \dots, n_k be non-negative integers. Denote by $G(m_1, \dots, m_k)$ the universal polynomial computing the number of curves having precisely n_i singularities of type T_i and no other singularities. There are then power series $B_1, B_2, B_3, B_4 \in \mathbb{Q}[[x_1, \dots, x_k]]$, such that*

$$\sum G(n_1, \dots, n_k) x_1^{n_1} \dots x_k^{n_k} = B_1^{c_1^2(L)} B_2^{c_1(L)c_1(S)} B_3^{c_1^2(S)} B_4^{c_2(S)}.$$

The same statement holds when the T_i are topological types.

Proof. This follows from Proposition 6.2 and the proof of Proposition 7.2, using the fact that W is irreducible in both the analytic and the topological case. \square

7.1.1. Analytic Types. We treat first the case of analytic singularity types. Fix a smooth, projective, connected surface S , a line bundle L on S , and analytic singularity types T_1, \dots, T_k . We assume that L is N -very ample, where N will be taken to be sufficiently large at various points in the proof.

In order to associate a 0-dimensional subscheme to an analytic singularity type, we need the following lemma, which states roughly that a singularity (C, p) is of analytic type T if it looks like a singularity of type T to M -th order, where M depends only on T .

Lemma 7.4. *Let (C, p) be a curve singularity of analytic type T . There is a positive integer M , depending only on T , such that if (C', p') is a curve singularity, the analytic type of (C', p') is T if and only if $\mathcal{O}_{C,p}/\mathfrak{m}^M \cong \mathcal{O}_{C',p'}/\mathfrak{m}^M$.*

Proof. This follows from [GLS, Cor. 2.24]. Note that the reference shows we can take $M = \tau + 2$, where τ is the Tjurina number of the singularity. \square

To a curve singularity type T we now associate a punctual geometric subscheme $W(T) \subset \text{Hilb}_0^k(\mathbb{C}^2)$ as follows. Let (C, p) be a germ of type T , and let M be the integer given by Lemma 7.4. Suppose the length of $\mathcal{O}_{C,p}/\mathfrak{m}_p^M$ is m . Let $W(T) \subset \text{Hilb}_0^m(\mathbb{C}^2)$ be the set of subschemes $Z \in \text{Hilb}_0^m(\mathbb{C}^2)$ with $Z \cong \text{Spec } \mathcal{O}_{C,p}/\mathfrak{m}^M$.

Let m_i be the integer such that $W(T_i) \subset \text{Hilb}_0^{m_i}(\mathbb{C}^2)$, for $i = 1, \dots, k$. Set $m = \sum m_i$, and define $W \subseteq S^{[m]}$ to be the set of subschemes of the form $Z_1 \sqcup \dots \sqcup Z_k$, where Z_i is isomorphic to a point in $W(T_i)$ for every i .

It is clear that W is a geometric subset, and in the notation of Section 2.3 we have $W = P((W(T_i)))$. We let $d := m - \dim W$, this is then the sum of the expected codimensions of the singularities T_1, \dots, T_k .

Note that $W(T_i)$ is irreducible and locally closed, as it is the orbit of a given point in $\text{Hilb}_0^{m_i}(\mathbb{C}^2)$ under the action of the connected algebraic group $\text{Aut}(\mathcal{O}_{\mathbb{C}^2,0}/\mathfrak{m}_i^M)$. It follows that W is irreducible and locally closed.

Lemma 7.5. *Let $Y \subset S^{[m]}$ be an irreducible, locally closed subset, and assume L is $(m-1)$ -very ample.*

- (i) *Let $\mathcal{Z} \subset S^{[m]} \times |L|$ denote the incidence locus of pairs (Z, C) with $Z \in Y$ and $Z \subset C$. We have $\dim \mathcal{Z} = \dim |L| + \dim Y - m$.*
- (ii) *Let $e = m - \dim Y$, and let $\mathbb{P}^e \subset |L|$ be a general subsystem. The number of pairs (Z, C) such that $Z \in Y$, $C \in \mathbb{P}^e$ and $Z \subset C$ is equal to*

$$\deg c_{\dim Y}(L^{[m]}) \cap [\overline{Y}]$$

Proof. (i) For any $Z \in Y$, the fibre of $\mathcal{Z} \rightarrow Y$ over Z is the projectivization of the kernel of $H^0(S, L) \rightarrow H^0(Z, L|_Z)$. By the $(m-1)$ -very ampleness of L , this homomorphism is surjective, so $\mathcal{Z} \rightarrow Y$ is a projective space bundle with fibres of dimension $|L| - m$. The claim follows.

(ii) See the proof of [Gö, Prop 5.2]. \square

Applying Lemma 7.5 (ii) with $W = Y$, the following lemma now concludes the proof of Proposition 7.2 in the analytic case.

Lemma 7.6. *Let $\mathbb{P}^d \subset |L|$ be a general subsystem, and assume L is N -very ample. Suppose (Z, C) is a pair such that $Z \in W$, $C \in \mathbb{P}^d$ and $Z \subset C$. Then C has k singularities of analytic types T_i , and C contains no other point of W .*

Proof. Let \mathbb{P}^d, C and Z be as in the statement of the lemma, and suppose $Z = \sqcup Z_i$, where Z_i is supported at $x_i \in C$ and where Z_i is isomorphic to a point in $W(T_i)$. We show the following claims: (1) That C has precisely k singularities, (2) that C has a singularity of type T_i at x_i , and (3) that C contains precisely one $Z \in W$.

(1): Clearly, C has at least k singularities. If C has more than k singularities, it contains a subscheme of the form $Z \sqcup Z'$, where Z' is defined by an ideal \mathfrak{m}_x^2 for some $x \in S$ where C is singular. The geometric set

$$W' := \{Z \sqcup Z' \mid Z \in W \text{ and } Z' = \text{Spec } \mathcal{O}_{S,x}/\mathfrak{m}_x^2\} \subset S^{[m+3]}$$

has dimension 2 greater than W . By Lemma 7.5 (i), we see that the set of $C \in |L|$ containing an element of W' has codimension $> d + 1$ in $|L|$ if L is $(m+2)$ -very ample. As $C \in \mathbb{P}^d \subseteq |L|$ is general, this proves the claim.

(2): Suppose for a contradiction that the singularity of C at x_1 is $T'_1 \neq T_1$. Let M be the integer associated to T_1 as in Lemma 7.4, and let $R = \mathcal{O}_{S,x_1}/\mathfrak{m}^M$.

As $T'_1 \neq T_1$, we have $Z_1 \subsetneq C \cap \text{Spec } R$. Let $f, g \in R$ be defining equations of Z_1 and $C \cap \text{Spec } R$ in R , we then have $(g) \subsetneq (f)$. This implies that $(g) \subseteq \mathfrak{m} \cdot (f) \subseteq \mathfrak{m}^{\text{ord}(f)+1} \cap (f)$, where $\text{ord}(f)$ is the maximal integer such that $f \in \mathfrak{m}^{\text{ord}(f)}$.

Hence C contains a subscheme of the form

$$Z'_1 \sqcup Z_2 \sqcup \dots \sqcup Z_k,$$

where $Z'_1 = \text{Spec } R/\mathfrak{m}^{\text{ord}(f)+1} \cup Z_1$. Let W' be the set of subschemes which can be written in this way. Then W' is geometric and has dimension $\leq \dim W$. Informally, this is because there is a surjection from W to W' defined by $Z \mapsto Z \cup \text{Spec } \mathcal{O}_{S, x_1}/\mathfrak{m}^{\text{ord}(f)+1}$.

It is not entirely obvious that this is a morphism, and we argue instead as follows. Let $W(T_1)'$ be the punctual geometric subset of schemes of the form $Z'_1 = \text{Spec } R/\mathfrak{m}^{\text{ord}(f)+1} \cup Z_1$, we see that both $W(T_1)$ and $W(T_1)'$ are orbits of an action of the group $\text{Aut}(\mathcal{O}_{\mathbb{C}^2, 0}/\mathfrak{m}^{m'_1})$, where m'_1 is the length of subschemes in $W(T_1)'$. The stabilizer of a point in $W(T_1)$ then contains the stabilizer of the corresponding point in $W(T_1)'$, and it follows that $\dim W' \leq \dim W$.

Let m' be the length of the points of W' . Clearly, we have $m' > m$, so we have

$$m' - \dim W' \geq m + 1 - \dim W' > m - \dim W = d.$$

As L is N -very ample, a slight modification of Lemma 7.5 (i) shows that the codimension of the locus of points containing a point from W' is $> d$, if $N \geq m + 1$. As $C \in \mathbb{P}^d$ for a general \mathbb{P}^d , it is not contained in this locus.

(3): By (1) and (2), we know that C has a singularity of type T_i at x_i , and suppose for a contradiction that there is a $Z' \in W$ with $Z' \subset C$ such that $Z' \neq Z$. Let $Z' = \sqcup Z'_i$ with Z'_i supported at x_i . Assume that $Z_i \neq Z'_i$ as subschemes. But by part (2), the singularity type associated to Z'_i must be T_i , and hence we have

$$Z_i = \text{Spec } \mathcal{O}_{C, x_i}/\mathfrak{m}^M = Z'_i,$$

where M is as in Lemma 7.4 for type T_i . □

7.1.2. Topological Singularities. We now turn to the case of topological singularities. Let S , L and N be as before, and fix topological singularity types T_1, \dots, T_k .

For any topological singularity type T , let $m = \deg(T)$ as defined in [KP]. We take $W(T) \subseteq \text{Hilb}_0^m(\mathbb{C}^2)$ to be defined in the same way as the subset denoted $H(D) \subseteq S^{[m]}$ in [KP, p.225], with $D = T$. Note that $H(D)$ is a subset of the full Hilbert scheme; we take its intersection with the set of subschemes supported at a given point to get a subset of the punctual Hilbert scheme. We briefly review the definition and refer to [KP] for further details.

Let (C, x) be a curve singularity, where $C \subset S$. We define the *configuration* \mathcal{C} associated to (C, x) as a certain subset of the set of infinitely near points in S over x contained in the strict transform of C , together with an integer weight at each point, equal to the multiplicity of the strict transform of C at that point. Specifically, we let \mathcal{C} be the set of *essential* points lying over x . The configuration \mathcal{C} then determines the Enriques diagram of (C, x) , which is equivalent to knowing the topological type.

We define a partial ordering on the set of configurations by saying that $\mathcal{C} \leq \mathcal{C}'$ if every point of \mathcal{C} is contained in \mathcal{C}' and has weight at least as big in \mathcal{C}' as it has in \mathcal{C} .

If \mathcal{C} is a configuration at $x \in S$, there is an associated ideal $\mathcal{I}_{\mathcal{C}}$ defining a subscheme $Z_{\mathcal{C}}$ supported at x , with the property that if (C, x) is a curve singularity with configuration \mathcal{C}' , then $\mathcal{C} \leq \mathcal{C}'$ if and only if $Z_{\mathcal{C}} \subset C$. The length m of $Z_{\mathcal{C}}$ depends only on the type T associated to \mathcal{C} , and we define the subset $W(T) = \{Z_{\mathcal{C}}\} \subset \text{Hilb}_0^m(\mathbb{C}^2)$, where \mathcal{C} runs over all configurations of type T at $0 \in \mathbb{C}^2$.

We define $W = W((T_i))$ similarly to in the analytic case. Note that $W((T_i))$ is locally closed [KP, p.225] and irreducible [KPT, Cor 5.8].

The following lemma shows roughly that containing a $Z \in W$ is generically equivalent to having the appropriate singularity types.

Lemma 7.7. *Let $Z \in W$, and let $|\mathcal{I}_Z L| \subset |L|$ be the subsystem of curves containing Z . A general $C \in |\mathcal{I}_Z L|$ has k singularities of types T_1, \dots, T_k .*

Proof. Let $\mathcal{I} = \mathcal{I}_Z$. By the discussion preceding [KP, Prop 3.5] it suffices to show that $H^0(\mathcal{I}L)$ generates the sheaf $\mathcal{I}L$. Let $x \in S$ be a point, then by Nakayama's Lemma it suffices to show that the vector space homomorphism $H^0(\mathcal{I}L) \rightarrow \mathcal{I}L/\mathfrak{m}_x \mathcal{I}L$ is surjective.

Let m_i be such that $W(T_i) \subset \text{Hilb}_0^{m_i}(\mathbb{C}^2)$. The length of the restriction of $Z(\mathcal{I})$ to x is $\leq \max m_i$, hence the colength of $\mathfrak{m}_x \mathcal{I}$ is $\leq m + 2 \max m_i$. If L is N -very ample and $N \geq m - 1 + 2 \max m_i$, the vector space map $H^0(L) \rightarrow L/\mathfrak{m}_x \mathcal{I}L$ is surjective. If $s \in H^0(L)$ lands in $\mathcal{I}L/\mathfrak{m}_x \mathcal{I}L \subseteq L/\mathfrak{m}_x \mathcal{I}L$, we have $s \in H^0(\mathcal{I}L)$. The claim follows. \square

After applying Lemma 7.5 (ii) with W as Y , we are reduced to showing the following.

Lemma 7.8. *Let $\mathbb{P}^d \subset |L|$ be a general subsystem, and assume L is N -very ample. Suppose (Z, C) is a pair such that $Z \in W$, $C \in \mathbb{P}^d$ and $Z \subset C$. Then C has exactly k singularities of topological types T_i , and C contains no other point of W .*

Proof. We let \mathcal{Z} be the incidence locus of pairs (Z, C) with $Z \in W$, $C \in |L|$ and $Z \subset C$. Let $p : \mathcal{Z} \rightarrow |L|$ and $q : \mathcal{Z} \rightarrow W$ be the projections. Let \mathbb{P}^d , C and Z be as in the statement of the lemma, and suppose that $Z = \sqcup Z_i$ with Z_i supported at $x_i \in C$. As in Lemma 7.6, we show (1) that C has k singularities, (2) that C has type T_i at x_i and (3) that C contains precisely one $Z \in W$.

(1): See the proof of Lemma 7.6.

(2): Assume for a contradiction that C has a singularity of type $T'_1 \neq T_1$ at x_1 . Define $W' \subset S^{[m']}$ in the same way as W , but replacing T_1 with T'_1 . Let $\mathcal{Z}' \subset |L| \times W'$, $p' : \mathcal{Z}' \rightarrow |L|$ and $q' : \mathcal{Z}' \rightarrow W'$ be the incidence locus and projections.

If D is a reduced curve which contains a $Z' \in W'$, it does not have types T_i . To see this, note first that $\deg(T'_1) > \deg(T_1)$, where the degree of a singularity type is as defined in [KP]. Hence the sum of the degrees of the singularities of D is bigger than $\sum \deg(T_i)$.

Consider now the set

$$X := p^{-1}(p'(\mathcal{Z}')) \subseteq \mathcal{Z}.$$

By Lemma 7.7 and the previous paragraph, $X \cap q^{-1}(Z)$ has positive codimension in $q^{-1}(Z)$ for every $Z \in W$. Thus X has positive codimension in \mathcal{Z} . We have $\dim \mathcal{Z} = \dim |L| - d$, hence we find that $p(X)$ has codimension $> d$ in $|L|$.

As there are at most finitely many alternatives for the type T'_1 appearing in $|L|$, there are at most finitely many closed subsets like $p(X)$ in $|L|$ to avoid. Since $C \in \mathbb{P}^d$ with \mathbb{P}^d general the claim follows.

(3): By (1) and (2), C has a singularity of type T_i at x_i and no other singularities. We may assume that Z is the subscheme defined by the configuration associated to the singularities of C . Suppose now for a contradiction that $Z' \in W$ is such that $Z' \subset C$ and $Z \neq Z'$. Let Z'_i be the component of Z' supported at x_i , then by (2) we see that the type associated to Z'_i is T_i . Let \mathcal{C} be the configuration associated to C at x_i , and let \mathcal{C}' be the configuration associated to Z'_i . Then $\mathcal{C}' \leq \mathcal{C}$, but as their Enriques diagrams are the same, we must have $\mathcal{C} = \mathcal{C}'$, and thus $Z'_i = Z_i$. \square

7.2. General Hypersurface Singularities. Without any extra work, the above extends to counts of analytic types of isolated singularities of hypersurfaces. Let (D, x) be the pair of a divisor D in a nonsingular variety X and an isolated singular point of D . We define the analytic type of the singularity (D, p) to be the isomorphism type of the complete local ring $\widehat{\mathcal{O}}_{D, x}$.

The exact same proof as in the surface case shows the following proposition. Note in particular that we do not need the nonsingularity of $X^{[n]}$ anywhere in the argument.

Proposition 7.9. *Let X be a smooth, projective, connected variety, let L be a line bundle on X , and let T_1, \dots, T_k be analytic singularity types. Suppose L is N -very ample, where N is an integer depending on the types T_i , and let d be the sum of the codimensions of the T_i . There is a rational polynomial G in the Chern numbers of (X, L) , depending only on the T_i , such that in a general $\mathbb{P}^d \subseteq |L|$ the number of divisors having precisely k isolated singularities of types T_i is given by G .*

There is a similar corollary for the generating function of these universal polynomials as in the curve case; we leave the statement of this to the reader.

7.3. BPS Spectrum Loci. Let C be a reduced, complete, locally planar algebraic curve, and consider the generating function

$$H_C(q) = \sum_{k=0}^{\infty} \chi(C^{[k]}) q^k.$$

Let the arithmetic and geometric genus of C be $g(C)$ and $\overline{g}(C)$, respectively. In [PT] it is shown that there are integers $n_{i,C}$, with $n_{i,C} = 0$ unless $\overline{g} \leq i \leq g$, such that

$$(3) \quad H_C(q) = \sum_{i=\overline{g}(C)}^{g(C)} n_{i,C} q^{g-i} (1-q)^{2i-2}.$$

For our purposes, it will be convenient to work with the index-shifted integers $m_{i,C} := n_{g-i,C}$. We define the BPS spectrum of C to be the sequence of integers $(m_{i,C})_{i=0}^{\infty}$. By the above, we have $m_{i,C} = 0$ if $i \geq g - \overline{g}$. If C has precisely k singularities of analytic type T_1, \dots, T_k , then stratifying $C^{[k]}$ it is not hard to show that the BPS spectrum of C depends only on the T_i .

By this observation, one may define the BPS spectrum of an analytic singularity type T as the BPS spectrum of a complete, reduced curve having one singularity of type T . The BPS spectrum of a singularity T is conjectured by Oblomkov and Shende in [OS] to be determined explicitly by the Milnor number and the HOMFLY polynomial of the link of T . In particular, the BPS spectrum of a curve should depend only on the topological types of the singularities of the curve.

Proposition 7.10. *Let S be a smooth, projective, connected surface, and let L be a line bundle on S . Let $m = (m_i)_{i=0}^{\infty}$ be a BPS spectrum, and denote by $|L|_m \subseteq |L|$ the locus of curves with BPS spectrum m . Let k be a nonnegative integer, let $\mathbb{P}^k \subseteq |L|$ be a general linear subsystem, and suppose L is N -very ample, where N is some integer depending on k and m . Then, there exists a rational polynomial G in 4 variables, depending only on k and m , such that*

$$\chi(\mathbb{P}^k \cap |L|_m) = G(c_1(L)^2, c_1(L)c_1(S), c_1(S)^2, c_2(S)).$$

If in addition it is known that $\mathbb{P}^k \cap |L|_m$ is 0-dimensional, this implies an enumerative result of the kind found in the previous subsection. This is essentially the

argument used in [KST] to compute the number of δ -nodal curves and prove the Göttsche Conjecture.

As in the previous sections, we will not give a specific value for the ampleness bound N .

The rest of this section contains the proof of Proposition 7.10.

Lemma 7.11. *Let k be an integer. There is a finite set of BPS spectra $m_1, \dots, m_{r(k)}$ such that if L is k -very ample and $\mathbb{P}^k \subset |L|$ is a general linear subsystem, then for every curve $C \in \mathbb{P}^k$ the BPS spectrum of C is among the m_i .*

Proof. If L is k -very ample, by [KST, Prop. 2.1], a curve C appearing in a generic \mathbb{P}^k is reduced and has geometric genus $\bar{g}(C) \geq g(C) - k$. By (3), the BPS spectrum of C is then determined by $\chi(C^{[i]})$ for $1 \leq i \leq k$.

Denote by $\text{Hilb}_p^j(C) \subset C^{[j]}$ the set of subschemes supported at $p \in C$. Stratifying $C^{[i]}$, we see that $\chi(C^{[i]})$ is determined by $\chi(C)$ and the integers $\chi(\text{Hilb}_p^j(C))$, where p ranges over every singular point of C and $1 \leq j \leq i$. The claim now follows from the easily shown fact that if j is fixed, then $\chi(\text{Hilb}_p^j(C))$ can take only finitely many values for $p \in C$ a singular point of a planar curve. \square

Suppose now that m_1, \dots, m_r are the possible BPS spectra in a general $\mathbb{P}^k \subseteq |L|$ for a k -very ample L . By [KST, Prop. 2.1] every curve $C \in \mathbb{P}^k$ satisfies $g(C) - \bar{g}(C) \leq k$. By (3), the BPS spectrum of such a C is determined by the k integers

$$\chi(C^{[1]}), \dots, \chi(C^{[k]})$$

and $g(C)$. For $i = 1, \dots, r$, there exists a polynomial $F_i \in \mathbb{Q}[g, x_1, \dots, x_k]$ such that

$$F_i \left(g(C_j), \chi(C_j), \dots, \chi(C_j^{[k]}) \right) = \delta_{ij}$$

when C_j is a curve with BPS spectrum m_j .

Let now $\mathbb{P}^k \subset |L|$ be general, let $\mathcal{C} \rightarrow \mathbb{P}^k$ be the family of curves, and let $\mathcal{C}^{[i]}/\mathbb{P}^k$ denote the relative Hilbert scheme. Every monomial m in the variables x_i determines a scheme $\mathcal{C}(m)$ by taking

$$\mathcal{C}(x_i) = \mathcal{C}^{[i]}/\mathbb{P}^k$$

and extending this by the rule

$$\mathcal{C}(m_1 \cdot m_2) = \mathcal{C}(m_1) \times_{\mathbb{P}^k} \mathcal{C}(m_2).$$

It is clear that

$$\begin{aligned} \chi(\mathcal{C}(m)) &= m(\chi(\mathcal{C}^{[1]}/\mathbb{P}^k), \dots, \chi(\mathcal{C}^{[k]}/\mathbb{P}^k)) \\ &= \sum_{i=1}^r \chi(|L|_{m_i} \cap \mathbb{P}^k) m \left(\chi(C_i^{[1]}), \dots, \chi(C_i^{[k]}) \right). \end{aligned}$$

Write F_i in the form

$$F_i = \sum_m f_m(g) m(x_1, \dots, x_k)$$

with m a monomial in the x_i and f_m a rational polynomial. Using the fact that $g(C) = (K_S L + L^2)/2 + 1$, we get

$$\chi(|L|_{m_i} \cap \mathbb{P}^k) = \sum_m f_m((K_S + L^2 + 1)) \chi(\mathcal{C}(m)).$$

The lemma below then finishes the proof of Proposition 7.10.

Lemma 7.12. *Let L be a line bundle, and let $\mathbb{P}^k \subseteq |L|$ be a general linear subsystem. Let $\mathcal{C} \rightarrow \mathbb{P}^k$ be the universal family of curves, and denote by $\mathcal{C}^{[i]} \rightarrow \mathbb{P}^k$ the relative Hilbert scheme of i points for this morphism. Then, the Euler characteristic*

$$\chi\left(\mathcal{C}^{[i_1]} \times_{\mathbb{P}^k} \cdots \times_{\mathbb{P}^k} \mathcal{C}^{[i_l]}\right)$$

is computed by a universal polynomial, provided that L is $((\sum_j i_j) - 1)$ -very ample.

Proof. For notational simplicity, we treat the case where $l = 2$, the general case is essentially the same. The case $l = 1$ is simpler, see [KST].

Let $f_n : \mathcal{C}^{[n]} \rightarrow S^{[n]}$ be the natural morphism. We show that there exists a finite stratification $S^{[n]}$ by geometric sets $W_{n,i}$ of universal type, such that

$$(4) \quad \chi(\mathcal{C}^{[k_1]} \times_{\mathbb{P}^k} \mathcal{C}^{[k_2]}) = \sum_{n=1}^{k_1+k_2} \sum_i i \cdot \chi(\mathcal{C}^{[n]} \cap f_n^{-1}(W_{n,i})).$$

Consider the function $g : S^{[k_1]} \times S^{[k_2]} \rightarrow \sqcup_{n=1}^{k_1+k_2} S^{[n]}$ defined pointwise by

$$g(Z_1, Z_2) = Z_1 \cup Z_2,$$

where the union is in the scheme-theoretic sense.

Define

$$W_{n,i} = \{Z \in S^{[n]} \mid \chi(g^{-1}(Z)) = i\}.$$

It is not hard to show that $W_{n,i}$ is geometric. Using the fact that $Z_1, Z_2 \subset C \Leftrightarrow Z_1 \cup Z_2 \subset C$, we also see that $W_{n,i}$ satisfies (4). Lemma 7.13 now completes the proof. \square

Lemma 7.13. *Let W be a geometric subset of $S^{[n]}$, and let $\mathbb{P}^k \subset |L|$ be a general linear subsystem, with L an $(n-1)$ -very ample line bundle. Let $\mathcal{C} \rightarrow \mathbb{P}^k$ be the family of curves, let $\mathcal{C}^{[n]}$ be the relative Hilbert scheme of the family, and let $f : \mathcal{C}^{[n]} \rightarrow S^{[n]}$ be the natural morphism. There exists a universal polynomial in the Chern numbers of (S, L) computing $\chi(f^{-1}(W) \cap \mathcal{C}^{[n]})$.*

Proof. The inclusion-exclusion principle for χ let us reduce to the case where W is closed and irreducible. Consider the diagram

$$\begin{array}{ccc} f^{-1}(W) \cap \mathcal{C}^{[n]} & \xrightarrow{f} & W \\ \downarrow & & \\ & & \mathbb{P}^k. \end{array}$$

The fibres of f are all projective spaces, since for a point $Z \in W$, the fibre over Z is the linear system of curves containing Z . Hence we have $\chi(f^{-1}(Z)) = \dim f^{-1}(Z) + 1$. Let $W_m = \{Z \mid \chi(f^{-1}(Z)) = m\}$. On W , consider the surjective homomorphism

$$H^0(S, L) \otimes \mathcal{O}_{S^{[n]}} \rightarrow L^{[n]},$$

let $E \subseteq H^0(S, L)$ be the $(k+1)$ -dimensional subspace defining \mathbb{P}^k , and let $\phi : E \otimes \mathcal{O}_W \rightarrow L^{[n]}$ be the induced homomorphism. Then

$$W_m = \{Z \in W \mid \dim \ker \phi = m\},$$

in other words $W_m = D_{k+1-m}(\phi) \setminus D_{k+1-m-1}(\phi)$, where $D_r(\phi)$ denotes the locus over which ϕ has rank $\leq r$. It thus suffices to compute $\chi(D_r(\phi))$ for all r .

By [PP, Thm 2.10], there exists a formula for the Euler characteristic of $D_r(\phi)$ as a polynomial in the Chern classes of $L^{[n]}$ capped with $c_{\text{SM}}(W)$, assuming the homomorphism $E \rightarrow L^{[n]}$ is r -general in the sense of [PP]. Choosing a Whitney stratification of W , r -generality amounts to saying that over each stratum, the

section of $\mathcal{H}om(E, L^{[n]})$ intersects the tautological degeneracy locus $D_r \setminus D_{r-1} \subseteq \mathcal{H}om(E, L^{[n]})$ transversely.

The $(n-1)$ -very ampleness of L implies there is a surjection $H^0(S, L) \otimes \mathcal{O}_W \rightarrow L^{[n]}$, inducing a morphism

$$W \rightarrow \mathrm{Gr}(H^0(S, L), n).$$

Choosing a subspace $E \subseteq H^0(S, L)$, the intersection of ϕ with $D_r \setminus D_{r-1}$ corresponds to the intersection of W with a certain smooth subset of $\mathrm{Gr}(H^0(S, L), n)$. By the Kleiman-Bertini transversality [Kl], for a general $E \subseteq H^0(S, L)$, the intersection of each Whitney stratum of W with this set will be smooth of the expected dimension. This shows that $E \otimes \mathcal{O}_W \rightarrow L^{[n]}$ is r -general in the sense of [PP].

Hence the formula of [PP, Thm 2.10] applies, and by Theorem 1.1 (ii), the statement of the lemma follows. \square

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